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ECE 45 Discussion 8 Notes

Impulse Function

The impulse function is an unusual function we use in linear system analysis because of its numerous useful properties. We cannot create a true impulse in real life (since it would require infinite magnitude at an infinitely precise point in time), but we can approximate it well enough to be able to apply its mathematical model to analyze real-life systems.

The impulse function is defined in the follow way:

$$\delta(t) = \frac{du(t)}{dt} = \begin{cases} \text{"}\infty\text{"} & \text{for} \quad t = 0 \\ 0 & \text{else} \end{cases} \quad \text{and} \quad \int_a^b \delta(t)\,dt = \begin{cases} 1 & \text{for} \quad a < 0 < b \\ 0 & \text{else} \end{cases}$$

i.e. its height at time $t = 0$ is infinite but is zero elsewhere, and the area under the curve is 1.

Note that by evaluating the second equation when $a = -\infty$ and $b = t$, we have

$$u(t) = \int_{-\infty}^{t} \delta(\tau)\,d\tau$$

which makes sense, since

$$\delta(t) = \frac{d}{dt}u(t).$$

i.e. the slot of $u(t)$ is zero everywhere, except at 0, where it is infinite.

It may seem that multiplication is not well-defined for the delta function, since its height is infinite, but multiplying the delta function by a constant will change its area (i.e. the value of its integral).

$$\text{That is:} \quad \int_a^b C\,\delta(t)\,dt = C\int_a^b \delta(t)\,dt = \begin{cases} C & \text{for} \quad a < 0 < b \\ 0 & \text{else} \end{cases}$$

This area-scaling property gives us the following relationship:

$$\int_a^b x(t)\,\delta(t - t_0)\,dt = \int_a^b x(t_0)\,\delta(t - t_0)\,dt = \begin{cases} x(t_0) & \text{for} \quad a < t_0 < b \\ 0 & \text{else} \end{cases}$$

The delta function is non-zero at a single point in time t_0, so the product $x(t)\,\delta(t - t_0) = 0$ for all $t \neq t_0$. At the instant $t = t_0$, the value of $x(t)$ is $x(t_0)$, so it acts as a scalar multiplier for the area. This implies

$$\mathcal{F}(\delta(t)) = \int_{-\infty}^{\infty} \delta(t)e^{-j\omega t}\,dt = e^0 \int_{-\infty}^{\infty} \delta(t)\,dt = 1$$

and similarly

$$\mathcal{F}^{-1}(\delta(\omega)) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \delta(\omega)e^{j\omega t} = \frac{1}{2\pi}$$

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Impulse Response

We have always dealt with LTI systems in terms of their transfer function $H(\omega)$. Another way to determine the transfer function of an LTI system is to input an impulse function to the system.

$$\delta(t) \longrightarrow H(\omega) \longrightarrow h(t)$$
$$h(t) = \mathcal{F}^{-1}(H(\omega))\,\mathcal{F}(\delta(t))\,) = \mathcal{F}^{-1}(H(\omega))$$

One way to determine the transfer function of a circuit/system is to send in a very brief, very high amplitude voltage/current/etc. "spike" into the circuit/system. This will produce some output $h(t)$. We could then take the Fourier transform of $h(t)$ to find $H(\omega)$. With very complicated circuits, this can be easier than calculating $H(\omega)$ analytically.

Example 1

What is $\delta(t)\,\sin(t)$ *and* $\int_{-\infty}^{\infty} \delta(t-\pi/4)\,\cos(t)\,dt$*?*

Solutions

$\delta(t) = 0$ for all $t \neq 0$ and $\sin(0) = 0$, so $\delta(t)\,\sin(t) = \begin{cases} (0)\,\sin(t) & \text{for } t \neq 0 \\ \delta(0)\,(0) & \text{for } t = 0 \end{cases}$

Thus $\delta(t)\,\sin(t) = 0$.

$\delta(t-\pi/4) = 0$ for all $t \neq \pi/4$ and $\cos(\pi/4) = 1/\sqrt{2}$, so

$$\int_{-\infty}^{\infty} \delta(t-\pi/4)\,\cos(t)\,dt = \frac{1}{\sqrt{2}}\int_{-\infty}^{\infty} \delta(t-\pi/4)\,dt = \frac{1}{\sqrt{2}}.$$

Example 2

If the impulse response of a system is $h(t) = \delta(t-4) - 2\,\delta(t) + \delta(t+4)$

(a) Determine $H(\omega)$ *as a purely real function.*

(b) Write the output $y(t)$ *in terms of* $x(t)$ *for arbitrary* $x(t)$*.*

(c) Let $x(t) = rect(t/8)$*. Write* $y(t)$ *as a piece-wise function and as a sum of unit step functions.*

Solutions

(a)

$$\begin{aligned} H(\omega) &= \mathcal{F}(h(t) = \mathcal{F}(\delta(t-4)) - 2\,\mathcal{F}(\delta(t)) + \mathcal{F}(\delta(t+4)) && \text{(by linearity property)} \\ &= \mathcal{F}(\delta(t))\,e^{-j\omega 4} - 2\,\mathcal{F}(\delta(t)) + \mathcal{F}(\delta(t))\,e^{j\omega 4} && \text{(by time-shifting property)} \\ &= -2 + e^{-j\omega 4} + e^{j\omega 4} = -2 + 2\,\cos(4\,\omega) \end{aligned}$$

(b) $x(t) \longrightarrow H(\omega) \longrightarrow y(t)$ so

$$\begin{aligned} Y(\omega) &= X(\omega)\,H(\omega) \\ &= X(\omega)\,(-2 + e^{-j\omega 4} + e^{j\omega 4}) \\ &= -2\,X(\omega) + e^{-j\omega 4}\,X(\omega) + e^{j\omega 4}\,X(\omega) \\ y(t) &= \mathcal{F}^{-1}(Y(\omega)) \\ &= -2\,\mathcal{F}^{-1}(X(\omega)) + \mathcal{F}^{-1}(e^{-j\omega 4}\,X(\omega)) + \mathcal{F}^{-1}(e^{j\omega 4}\,X(\omega)) && \text{(by linearity property)} \\ &= -2\,x(t) + x(t-4) + x(t+4) && \text{(by time-shifting property)} \end{aligned}$$

(c) Recall: $\text{rect}\,(t) = \begin{cases} 1 & \text{for} \quad -1/2 \leq t \leq 1/2 \\ 0 & \quad\text{else} \end{cases}$

So $x(t+4)$ is a rectangle of height 1, width 8, centered at -4.
$-2\,x(t)$ is a rectangle of height -2, width 8, centered at 0.
$x(t-4)$ is a rectangle of height 1, width 8, centered at $t=4$. Thus,

$$y(t) = \text{rect}\left(\frac{t+4}{8}\right) - 2\,\text{rect}\left(\frac{t}{8}\right) + \text{rect}\left(\frac{t-4}{8}\right) = \begin{cases} -1 & \text{for} \quad |t| \leq 4 \\ 1 & \text{for} \quad 4 < |t| \leq 8 \\ 0 & \text{for} \quad |t| > 8 \end{cases}$$

$$= u(t+8) - 2u(t+4) + 2u(t-4) - u(t-8).$$

Example 3

If the input to an LTI system is $x(t) = \dfrac{d}{dt}\Delta(t/5)$, *determine the output,* $y(t)$, *where the impulse response of the system is* $h(t) = u(t)$.

Solutions

Recall $\Delta(t) = \begin{cases} 1 - 2|t| & \text{for} \quad -1/2 \leq t \leq 1/2 \\ 0 & \quad\text{else} \end{cases}$

Let $f(t) = \Delta(t/5)$. Then by derivative property, $X(\omega) = j\omega\, F(\omega)$.

Since $x(t) \;\rightarrow\; H(\omega) \;\rightarrow\; y(t)$, we have

$$Y(\omega) = H(\omega)\, X(\omega) = H(\omega)\,(j\omega F(\omega)) = (j\omega H(\omega))\, F(\omega).$$

By derivative property, $\mathcal{F}^{-1}\left(j\omega\, H(\omega)\right) = \dfrac{d}{dt}u(t) = \delta(t)$, so $j\omega H(\omega) = 1$.

Thus $Y(\omega) = F(\omega)$ and $y(t) = f(t) = \Delta(t/5)$.

Alternatively, we could have calculated $X(\omega)$ and used it to calculate $Y(\omega)$, but instead we used the properties of the Fourier transform to save ourselves some work. Directly calculating $Y(\omega)$ and $y(t)$ is a good exercise.

Example 4:

Let a *be a real number. Suppose* $(t-a)u(t-a)$ *is the input to an LTI system with frequency response* $H(\omega) = j\omega$. *Find the output of such a system.*

Solutions

If $x(t)$ is the input to $H(\omega)$, then the Fourier transform of the output is given by $X(\omega)H(\omega)$, where $X(\omega)$ is the Fourier transform of $x(t)$. Thus $Y(\omega) = j\omega X(\omega)$, so by the time-derivative property, $y(t) = \frac{d}{dt}x(t)$.

Thus the output when $tu(t)$ is the input is:

$$\frac{d}{dt}(tu(t)) = 1u(t) + t\delta(t) = u(t).$$

Then by the time invariance of the system, when $(t-a)u(t-a)$ is the input, $u(t-a)$ is the output.