

## ECE 45 Discussion 9 Notes

### Convolution

Convolution is a mathematical operation defined by:

$$f(t) * g(t) = \int_{-\infty}^{\infty} f(\tau) g(t - \tau) d\tau$$

We are typically concerned with convolution in the time domain, but we can also do convolution in the frequency domain (or over any other variable):

$$F(\omega) * G(\omega) = \int_{-\infty}^{\infty} F(\theta) G(\omega - \theta) d\theta$$

### Convolution and the Fourier Transform

An interesting property of convolution relates the Fourier transforms of a convolution:

$$f(t) * g(t) \longleftrightarrow F(\omega) G(\omega)$$

i.e. Convolution in the time domain corresponds to multiplication in the frequency domain. Thus for  $x(t) \longrightarrow H(\omega) \longrightarrow y(t)$

$$y(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} Y(\omega) e^{j\omega t} d\omega = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega) H(\omega) e^{j\omega t} d\omega = x(t) * h(t)$$

In general, we like to use the Fourier transform to determine the output of a system, but oftentimes, we end up with something we do not know the inverse Fourier transform of, so we use the integration method.

### Convolution is Commutative

$$f(t) * g(t) = \int_{-\infty}^{\infty} f(\tau) g(t - \tau) d\tau = \int_{-\infty}^{\infty} g(\tau) f(t - \tau) d\tau = g(t) * f(t)$$

### Convolution is Distributive Over Addition

$$f(t) * (g(t) + h(t)) = f(t) * g(t) + f(t) * h(t)$$

### Time-Shifted Convolution

$$\text{If } y(t) = x(t) * h(t) \text{ then } x(t - t_0) * h(t) = x(t) * h(t - t_0) = y(t - t_0)$$

### Time-Derivative in Convolution

$$\text{If } y(t) = x(t) * h(t) \text{ then } \frac{dx(t)}{dt} * h(t) = x(t) * \frac{dh(t)}{dt} = \frac{dy(t)}{dt}$$

These (and other properties of convolution) can be verified by looking at the convolution in the Frequency domain.

### Example 1

Find  $y(t) = \delta(t - t_0) * x(t)$  for any  $x(t)$ .

### Solutions

Recall:  $\int_{-\infty}^{\infty} \delta(t - a) f(t) dt = \int_{-\infty}^{\infty} \delta(t - a) f(a) dt = f(a)$ , since  $\delta(t - a) = 0$  for all  $t \neq a$ . Then

$$\int_{-\infty}^{\infty} \delta(\tau - t_0) x(t - \tau) d\tau = \int_{-\infty}^{\infty} \delta(\tau - t_0) x(t - t_0) d\tau = x(t - t_0) \int_{-\infty}^{\infty} \delta(\tau) d\tau = x(t - t_0)$$

We could have also used the Fourier transform to calculate  $y(t)$ .

### Example 2

Find  $y(t) = u(t) * u(t)$

### Solutions

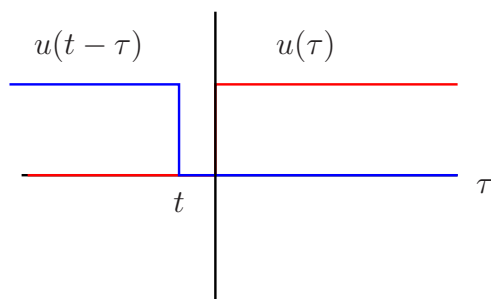
$$y(t) = \int_{-\infty}^{\infty} u(\tau) u(t - \tau) d\tau = \int_{-\infty}^0 0 u(t - \tau) d\tau + \int_0^{\infty} 1 u(t - \tau) d\tau = \int_0^{\infty} u(t - \tau) d\tau$$

When  $t \geq 0$  we have:

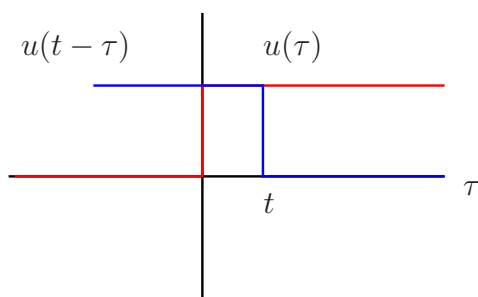
$$u(t - \tau) = \begin{cases} 1 & \tau \leq t \\ 0 & \tau > t \end{cases}$$

But if  $t < 0$ ,  $u(t - \tau) = 0$ , because we are integrating  $\tau$  from 0 to  $\infty$ , so we will never have  $\tau \leq t$  so we have:

$$y(t) = \begin{cases} \int_0^t 1 d\tau & t \geq 0 \\ 0 & t < 0 \end{cases} = \begin{cases} t & t \geq 0 \\ 0 & t < 0 \end{cases} = t u(t)$$



When  $t < 0$ , the functions  $u(\tau)$  and  $u(t - \tau)$  are never non-zero at the same values  $\tau$ , i.e. there is no “overlap.” This implies their product  $u(\tau)u(t - \tau)$  is zero for all values of  $\tau$ .



When  $t > 0$ , both functions  $u(\tau)$  and  $u(t - \tau)$  are non-zero when  $\tau$  is between 0 and  $t$ , so their product  $u(\tau)u(t - \tau)$  is non-zero in this region.

We also could have attempted this problem using the Fourier Transform

$$\mathcal{F}(u(t)) = \pi \delta(\omega) + 1/(j\omega) \rightarrow y(t) = \mathcal{F}^{-1}((\pi \delta(\omega) + 1/(j\omega))^2) = ??$$

Unfortunately, it is not something we can easily look up or determine, so it is easier to use the integration method in this case.

In general, convolution with the unit step function is relatively straightforward to compute. For any  $f(t)$ :

$$f(t) * u(t) = \int_{-\infty}^{\infty} f(\tau) u(t - \tau) d\tau = \int_{-\infty}^t f(\tau) d\tau.$$

For example, if  $f(t) = (t^2 + t + 1) u(t)$ , then

$$f(t) * u(t) = \int_{-\infty}^t (\tau^2 + \tau + 1) u(\tau) d\tau.$$

We have  $\tau < t$  in the integral, so when  $t < 0$ , we have  $\tau < 0$ ,  $u(\tau) = 0$ , which implies  $f(t) * u(t) = 0$ , when  $t < 0$ , and when  $t \geq 0$ , we have

$$\int_{-\infty}^t (\tau^2 + \tau + 1) u(\tau) d\tau = \int_0^t (\tau^2 + \tau + 1) d\tau = \frac{t^3}{3} + \frac{t^2}{2} + t$$

and so

$$f(t) * u(t) = \left( \frac{t^3}{3} + \frac{t^2}{2} + t \right) u(t).$$

## Example 2

Compute  $g(t) * h(t)$  where

$$g(t) = \begin{cases} -1 & 1 \leq t \leq 3 \\ 0 & \text{else} \end{cases} \quad h(t) = \begin{cases} 2t & 0 \leq t \leq 1 \\ 0 & \text{else} \end{cases}$$

## Solutions

We could attempt to use the Fourier transform, but we would quickly discover we end up with something we do not know how to take the inverse Fourier transform of. Instead, we will use the integration method:

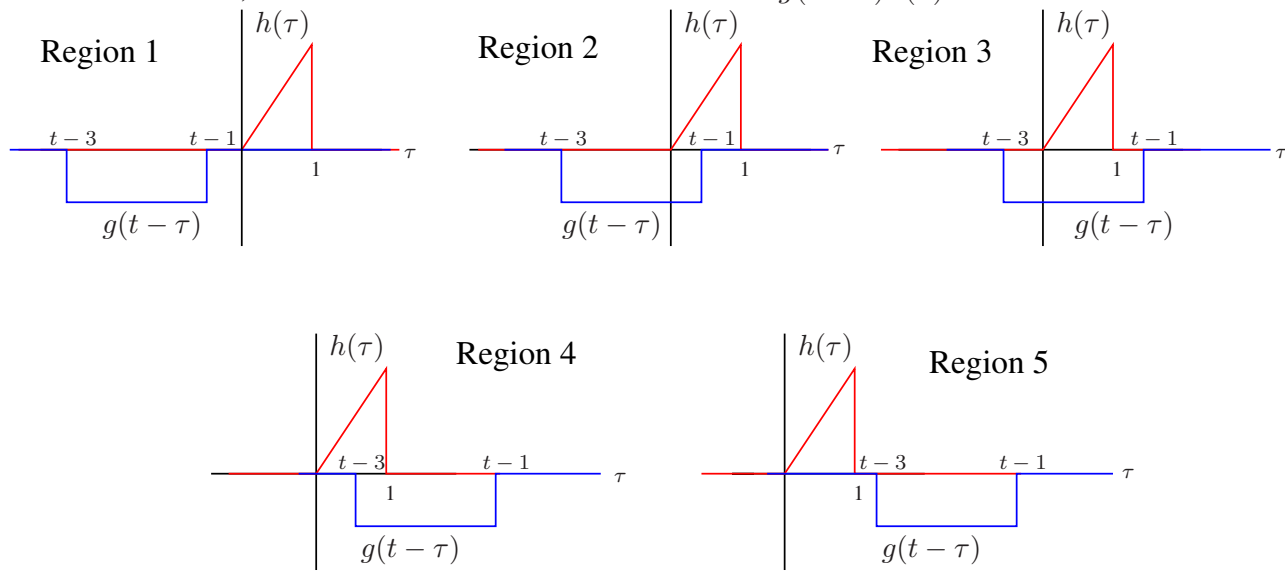
$$g(t) * h(t) = \int_{-\infty}^{\infty} h(\tau) g(t - \tau) d\tau$$

The next step is a bit tricky. Since  $g(t)$  and  $h(t)$  are both piece-wise functions, and  $g(t - \tau)$  will behave differently, depending on what values of  $t$  we select. We will have different regions of overlap (and thus different bounds on our integrals) depending on the values of  $t$  and  $\tau$ .

We know that

$$g(t - \tau) = \begin{cases} -1 & t - 3 \leq \tau \leq t - 1 \\ 0 & \text{else} \end{cases} \quad \text{and} \quad h(\tau) = \begin{cases} 2\tau & 0 \leq \tau \leq 1 \\ 0 & \text{else} \end{cases}$$

For each value of  $t$ , we want to find the values of  $\tau$  such that  $g(t - \tau)h(\tau)$  is non-zero.



The bounds on  $t$  become the bounds for the piece-wise function, and the bounds on  $\tau$  become the bounds on our integral.

- Region 1:  $t < 1$

No overlap, so in this region  $h(t) * g(t) = 0$

- Region 2:  $1 < t < 2$

Overlap for  $\tau = 0$  to  $t - 1$ , so in this region  $h(t) * g(t) = \int_0^{t-1} -2\tau d\tau = -(t-1)^2$

- Region 3:  $2 < t < 3$

Overlap for  $\tau = 0$  to  $1$ , so in this region  $h(t) * g(t) = \int_0^1 -2\tau d\tau = -1$

- Region 4:  $3 < t < 4$

Overlap for  $\tau = t - 3$  to  $1$ , so in this region  $h(t) * g(t) = \int_{t-3}^1 -2\tau d\tau = (t-3)^2 - 1$

- Region 5:  $t > 4$

No overlap, so in this region  $h(t) * g(t) = 0$

Hence

$$g(t) * h(t) = \begin{cases} 0 & t < 1 \\ -(t-1)^2 & 1 \leq t \leq 2 \\ -1 & 2 \leq t \leq 3 \\ (t-3)^2 - 1 & 3 \leq t \leq 4 \\ 0 & t > 4 \end{cases}$$