University of California, San Diego
ECE 45 Spring 2014
Midterm Exam 2 SOLUTIONS

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- No Books, No Notes, No calculators allowed

Question	Score
1	/25
2	/40
3	/20
4	/15
Total:	/100

Problem 1) [25 pts]

a. [20 pts] Determine and plot the magnitude and the phase of the Fourier Transform of the function:

$$h(t) = \delta(t) - \left(\cos\left(t\ \frac{\pi}{2}\right) - j\sin\left(t\ \frac{\pi}{2}\right)\right)\ \frac{\sin\left(t\frac{\pi}{4}\right)}{t\ \pi}$$

b. [5 pts] Assume $h(t)$ is the impulse response to an LTI system. Determine the output for the following inputs:

i. $x_1(t) = 2\cos\left(t\ \frac{2\pi}{3}\right)$

ii. $x_2(t) = \sin(4t) +\ 1$

a.

Note: $\left(\cos\left(t\,\frac{\pi}{2}\right) - j\sin\left(t\,\frac{\pi}{2}\right)\right) = e^{-jt\,\pi/2}$

Let $\quad g(t) = \frac{\sin\left(t\,\frac{\pi}{4}\right)}{t\,\pi}$

We can write $h(t)$ as: $\quad h(t) = \delta(t) - \ e^{-jt\,\pi/2}\, g(t)$

By F.T. pairs table: $\; G(j\omega) = \begin{cases} 1 & |\omega| \le \pi/4 \\ 0 & |\omega| > \pi/4 \end{cases}$

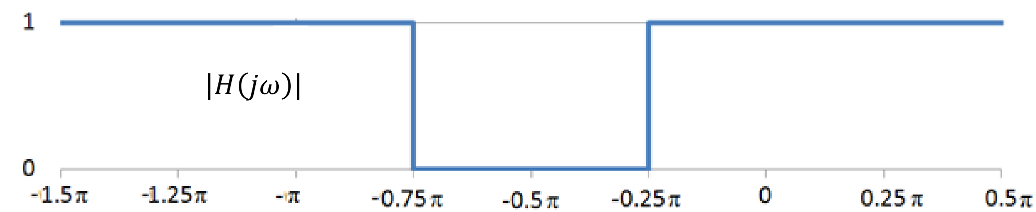
By Freq Shifting Property:

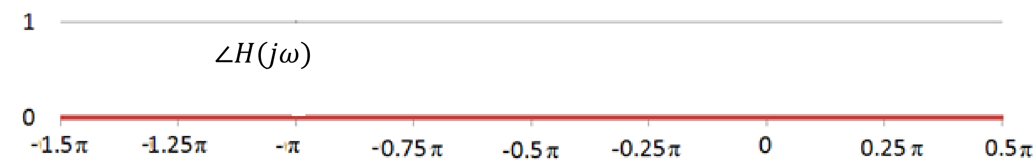
$e^{-jt\,\pi/2}\, g(t) \leftrightarrow G\left(j\left(\omega + \frac{\pi}{2}\right)\right) = \begin{cases} 1 & |\omega + \pi/2| \le \pi/4 \\ 0 & |\omega + \pi/2| > \pi/4 \end{cases}$

Thus by Linearity: $H(j\omega) = 1 - G\left(j\left(\omega + \frac{\pi}{2}\right)\right)$

$$H(j\omega) = \begin{cases} 0 & -3\pi/4 \le \omega \le -\pi/4 \\ 1 & else \end{cases}$$

$H(j\omega)$ is purely real, so $|\,H(j\omega)| = H(j\omega)$ and $\angle H(j\omega) = 0$





b.

i. $X_1(j\omega) = 2\pi\left[\delta\left(\omega - \frac{2\pi}{3}\right) + \ \delta\left(\omega + \frac{2\pi}{3}\right)\right]$

$$Y_1(j\omega) = \ X_1(j\omega)H(\omega) = \ 2\pi\,\delta\left(\omega - \frac{2\pi}{3}\right) \rightarrow y_1(t) = \ e^{j\,t\,\frac{2\pi}{3}}$$

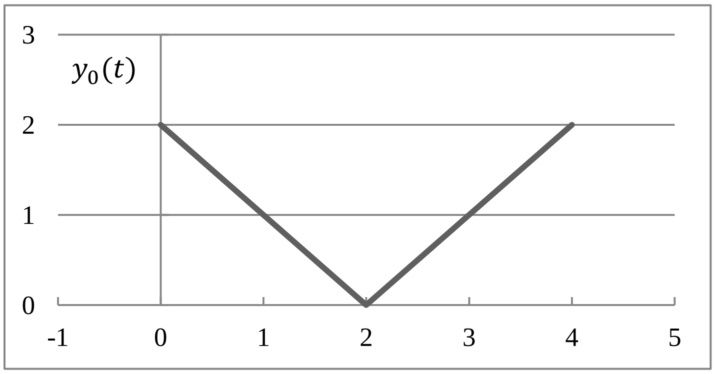
ii. $X_2(j\omega) = \frac{\pi}{j}[\delta(\omega - 4) - \ \delta(\omega + 4)] + \ 2\,\pi\,\delta(\omega)$

$$Y_1(j\omega) = \ X_1(j\omega)H(\omega) = \ \frac{\pi}{j}[\delta(\omega - 4) - \ \delta(\omega + 4)] + \ 2\,\pi\,\delta(\omega) \rightarrow y_1(t) = \ x_1(t)$$

Problem 2) [40 pts]

a. [20 pts] When the signal $x(t)$ is sent into the LTI system with transfer function $H_0(\omega)$, the output is $y_0(t)$. Determine the transfer function $H_0(\omega)$

$$x(t) = u(t+2) - 2u(t) + u(t-2)$$



b. [5 pts] Evaluate

$$\left(\int_{-\infty}^{\infty} |X(j\omega)|^2 \, d\omega\right)^2$$

c. [15 pts] $x(t)$ is sent into a new system with impulse response $h_1(t) = \delta(t-1) + \delta(t+1)$

Determine and plot the output $y_1(t)$

a.

$$x(t) = u(t+2) - 2\,u(t) + u(t-2) = \begin{cases} 1 & -2 \le t < 0 \\ -1 & 0 \le t < 2 \\ 0 & else \end{cases}$$

Let $f(t) = \begin{cases} t & -2 \le t < 0 \\ -t & 0 \le t < 2 \\ 0 & else \end{cases}$

Note that: $x(t) = \frac{df(t)}{dt}$ and $f(t) = -y_0(t+2)$ so $x(t) = -\frac{d\,y_0(t+2)}{dt}$

Thus by linearity, time shifting, and derivative properties:
$X(j\omega) = -j\omega\, e^{j\omega\, 2}\, Y_0(j\omega)$

$$H_0(\omega) = \frac{Y_0(j\omega)}{X(j\omega)} = \boxed{\frac{-e^{-j\omega\, 2}}{j\omega}}$$

b.

$$\left(\int_{-\infty}^{\infty} |X(j\omega)|^2\, d\omega\right)^2 = \left(2\pi \int_{-\infty}^{\infty} |x(t)|^2\, dt\right)^2$$

$$= \left(2\pi\left(\int_{-2}^{0} 1^2\, dt + \int_{0}^{2} (-1)^2\, dt\right)\right)^2 = \left(2\pi(2+2)\right)^2 = \boxed{64\,\pi^2}$$

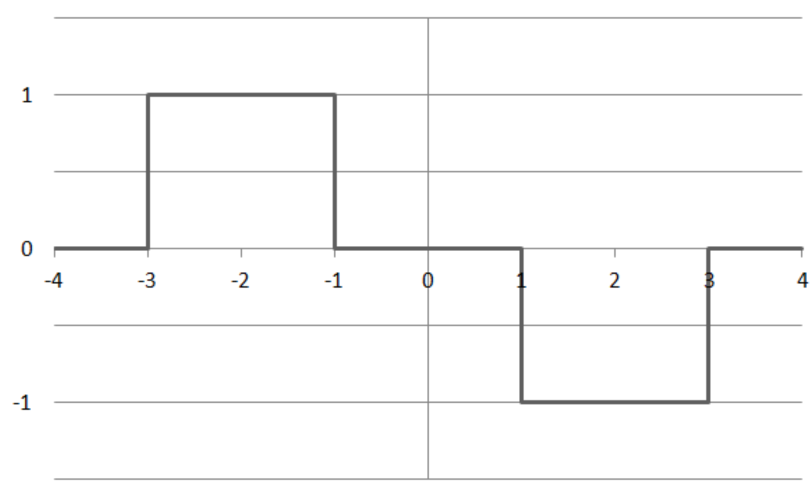
c.

$$h_1(t) = \delta(t-1) + \delta(t+1) \leftrightarrow H_1(\omega) = e^{-j\omega} + e^{j\omega}$$

$$Y_1(j\omega) = X(j\omega)H_1(\omega) = X(j\omega)e^{-j\omega} + X(j\omega)e^{j\omega}$$

$$y_1(t) = x(t-1) + x(t+1) = u(t+3) - u(t+1) - u(t-1) + u(t-3)$$

$$\boxed{y_1(t) = \begin{cases} 1 & -3 \le t < -1 \\ -1 & 1 \le t < 3 \\ 0 & else \end{cases}}$$



Problem 3) [20 pts]

a. [10 pts] If $F(j\omega) = B(j\omega)C(j\omega)$ and $G(j\omega) = \frac{e^{j\omega}}{2} B\left(\frac{j\omega}{3}\right) C\left(\frac{j\omega}{3}\right)$

Write $g(t)$ in terms of $f(t)$

b. **[10 pts]** Give an example of an input $x(t)$, a system $H(\omega)$, and an output $y_1(t)$ such that:

$$x(t) \rightarrow H(\omega) \rightarrow y_1(t) \quad \text{and} \quad x(2t) \rightarrow H(\omega) \rightarrow y_2(t) \neq y_1(2t)$$

(i.e. show that it is not generally true that the input $x(2t)$ has the output $y_1(2t)$)

a.

Let $G'(j\omega) = X\left(\frac{j\omega}{3}\right) H\left(\frac{j\omega}{3}\right) = F\left(\frac{j\omega}{3}\right)$

By time scaling property: $g'(t) = 3\, f(3t)$

$G(j\omega) = \frac{e^{j\omega}}{2} G'(j\omega)$

Thus: $g(t) = \frac{1}{2} g'(t+1) = \boxed{\frac{3}{2} f(3t+3)}$

b.

Consider $H(\omega) = \begin{cases} 1 & |\omega| \leq 1 \\ 0 & else \end{cases}$ and $x(t) = \cos\left(\frac{3}{4}t\right)$

$y_1(t) = \cos\left(\frac{3}{4}t\right)$ so $y_1(2t) = \cos\left(\frac{3}{2}t\right)$

but $y_2(t) = \left|H\left(\frac{3}{2}\right)\right| \cos\left(\frac{3}{2}t\right) = 0$

Thus $y_2(t) \neq y_1(2t)$

Problem 4) [15 pts]

Suppose the signal $g(t)$ is sent into a system $H(\omega) = (2 + j\omega)$

Unfortunately, we don't know what $g(t)$, but we do know that

$$G(j\omega) = j\frac{d}{d\omega}F(-j\omega) \quad \text{where } F(j\omega) = \frac{1}{2-j\omega}$$

Determine the output, $y(t)$, of the system
(Hint: You can do this without performing calculus)

Bonus: [5 pts] Determine the Fourier transform of

$$x(t) = \frac{1}{2 - jt}$$

$$F(-j\omega) = \frac{1}{2 + j\omega} \leftrightarrow f(-t) = e^{-2t}\, u(t)$$

$$G(j\omega) = j\frac{d}{d\omega}\, F(-j\omega) \;\leftrightarrow\; g(t) = t\, f(-t) = t\, e^{-2t}\, u(t)$$

$$G(j\omega) = \frac{1}{(2 + j\omega)^2}$$

$$Y(j\omega) = H(\omega)G(\omega) = \frac{2+j\omega}{(2+j\omega)^2} = \frac{1}{2+j\omega}$$

$$\boxed{y(t) = e^{-2t}u(t)}$$

Bonus:

Note: $x(t) = F(jt)$

$$X(j\omega) = \int_{-\infty}^{\infty} x(t)\; e^{-j\omega t} dt = \int_{-\infty}^{\infty} F(jt)\; e^{-j\omega t} dt = 2\pi f(-\omega) = \boxed{2\pi\, e^{-2\omega} u(\omega)}$$