University of California, San Diego
ECE 45 Spring 2016
Midterm Exam II

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Print your name: SOLUTIONS

Student ID Number: ____________________

Note: No books, notes, calculators, or other electronic devices allowed.

Question	Score
1	10 / 10
2	10 / 10
3	10 / 10
Total	30 / 30

Signal	Fourier Transform
$x(t)$	$X(j\omega)$
$y(t)$	$Y(j\omega)$
$ax(t) + by(t)$	$aX(j\omega) + bY(j\omega)$
$x(t - t_0)$	$e^{-j\omega t_0} X(j\omega)$
$e^{j\omega_0 t} x(t)$	$X(j(\omega - \omega_0))$
$x^*(t)$	$X^*(-j\omega)$
$x(-t)$	$X(-j\omega)$
$x(at)$	$\frac{1}{\|a\|} X\left(\frac{j\omega}{a}\right)$
$x(t) * y(t)$	$X(j\omega)Y(j\omega)$
$x(t)y(t)$	$\frac{1}{2\pi} X(j\omega) * Y(j\omega)$
$\frac{d}{dt} x(t)$	$j\omega X(j\omega)$
$tx(t)$	$j \frac{d}{d\omega} X(j\omega)$
$X(t)$	$2\pi x(-j\omega)$
1	$2\pi\delta(\omega)$
$\delta(t)$	1
$u(t)$	$\frac{1}{j\omega} + \pi\delta(\omega)$
$\cos \omega_0 t$	$\pi[\delta(\omega - \omega_0) + \delta(\omega + \omega_0)]$
$\sin \omega_0 t$	$\frac{\pi}{j}[\delta(\omega - \omega_0) - \delta(\omega + \omega_0)]$
$e^{-at}u(t)$, $\mathrm{Re}\{a\} > 0$	$\frac{1}{a + j\omega}$
$\begin{cases} 1, & \|t\| \leq T \\ 0, & \|t\| > T \end{cases}$	$\dfrac{2 \sin \omega T}{\omega}$
$\dfrac{\sin Wt}{\pi t}$	$\begin{cases} 1, & \|\omega\| \leq W \\ 0, & \|\omega\| > W \end{cases}$

Some Useful Formulas:

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\omega) e^{j\omega t} \, d\omega$$

$$X(j\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} \, dt$$

$$u(t) = \begin{cases} 1, & t \geq 0 \\ 0, & t < 0 \end{cases}$$

$$\delta(t) = \frac{d}{dt} u(t)$$

$$e^{j\theta} = \cos\theta + j \sin\theta$$

$$\cos\theta = \frac{e^{j\theta} + e^{-j\theta}}{2}$$

$$\sin\theta = \frac{e^{j\theta} - e^{-j\theta}}{2j}$$

$$\int_{-\infty}^{\infty} |x(t)|^2 \, dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} |X(j\omega)|^2 \, d\omega$$

$$e^{j\omega_0 t} \to H(\omega) \to H(\omega_0) e^{j\omega_0 t}$$

Problem 1: [10 points]

Suppose an LTI system has impulse response

$$h(t) = \frac{\sin(mt - ma\pi^2)}{t - a\pi^2}.$$

Determine the output of the system $y(t)$, when the input to the system is

$$x(t) = \cos\left(\frac{nt}{\pi}\right) + \sin\left(\frac{2nt}{\pi}\right).$$

Version 1: $m = 2$, $n = 6$, $a = 7$

Version 2: $m = 3$, $n = 9$, $a = 8$

Solution:

From the table we have,

$$\mathcal{F}\left(\frac{\sin(mt)}{t}\right) = \pi\,\mathcal{F}\left(\frac{\sin(mt)}{\pi t}\right) = \begin{cases} \pi & \text{if } |\omega| < m \\ 0 & \text{otherwise} \end{cases}$$

So by the time-shift property,

$$H(\omega) = \begin{cases} \pi e^{-j\omega a\pi^2} & \text{if } |\omega| < m \\ 0 & \text{otherwise} \end{cases} = \begin{cases} \pi e^{-j\pi^2 \omega a} & \text{if } |\omega| < m \\ 0 & \text{otherwise.} \end{cases}$$

In general,

$$\cos(\omega_0 t) \longrightarrow H(\omega) \longrightarrow |H(\omega_0)| \cos(\omega_0 t + \angle H(\omega_0))$$

We have $n/\pi < m$ and $2n/\pi > m$, so

$$\cos\left(\frac{n}{\pi} t\right) \longrightarrow H(\omega) \longrightarrow n \cos\left(\frac{n}{\pi} t + \pi/2 - na\pi\right)$$

and

$$\sin\left(\frac{2n}{\pi} t\right) \longrightarrow H(\omega) \longrightarrow 0$$

Thus by the linearity of the system,

$$y(t) = \pi \cos\left(\frac{n}{\pi} t - na\pi\right)$$

but in both cases na is even, so

$$y(t) = \pi \cos\left(\frac{n}{\pi} t\right)$$

Problem 2: [10 points]

Recall the energy of a signal $x(t)$ is given by $\int_{-\infty}^{\infty} |x(t)|^2\, dt$. Find the energy of

$$x(t) = \frac{\sin(m\pi t)}{\pi t}\cos(n\pi t)$$

Hint: Use Parseval's Theorem and the frequency shifting property.

Version 1: $m = 6$, $n = 7$

Version 2: $m = 4$, $n = 5$

Solution:

By Parseval's Theorem, we have

$$\int_{-\infty}^{\infty} |x(t)|^2\, dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} |X(\omega)|^2\, d\omega$$

Let $z(t) = \dfrac{\sin m\pi t}{\pi t}$. Then

$$x(t) = z(t)\cos(n\pi t) = \frac{e^{jn\pi t}z(t) + e^{-jn\pi t}z(t)}{2}$$

so by the frequency shifting property

$$X(\omega) = \frac{Z(\omega - n\pi) + Z(\omega + n\pi)}{2}$$

and by the table,

$$Z(\omega) = \begin{cases} 1 & \text{if } |\omega| < m\pi \\ 0 & \text{otherwise} \end{cases}$$

which implies

$$X(\omega) = \begin{cases} 1 & \text{if } (n-m)\pi < \omega < (n+m)\pi \\ & \text{or } (-n-m)\pi < \omega < (-n+m)\pi \\ 0 & \text{otherwise} \end{cases}$$

Then $|X(\omega)|^2 = \left(\dfrac{Z(\omega - n\pi) + Z(\omega + n\pi)}{2}\right)^2 = \dfrac{Z(\omega - n\pi) + Z(\omega + n\pi)}{4} = \dfrac{1}{2}X(\omega)$, so

$$\int_{-\infty}^{\infty} |x(t)|^2\, dt = \frac{1}{2\pi}\left(\int_{(n-m)\pi}^{(n+m)\pi} \frac{1}{4}\, d\omega + \int_{(-n-m)\pi}^{(-n+m)\pi} \frac{1}{4}\, d\omega\right) = \frac{4m\pi}{8\pi} = \frac{m}{2}$$

Problem 3: [10 points]

Suppose we are trying to transmit a signal through an LTI channel that behaves as an ideal low-pass filter with cut-off frequency m. That is, the frequency response of the channel $H(\omega)$ is equal to 1 for $\omega \in [-m, m]$ and is equal to 0 otherwise.

Which of the following signals will pass through the channel without any distortion? (i.e. the output is exactly equal to the input) Justify your answers.

[2 pts] $x_1(t) = e^{j2mt}$

[2 pts] $x_2(t) = \pi$

[2 pts] $x_3(t) = \text{sinc}\,(mt/2)$

[4 pts] $x_4(t) = (u(t) - u(t-b))\,e^{-at}$

Version 1: $m = 6$, $a = 7$, $b = 10$

Version 2: $m = 4$, $a = 5$, $b = 11$

Solution:

The described channel has frequency response

$$H(\omega) = \begin{cases} 1 & \text{if } |\omega| \leq m \\ 0 & \text{otherwise.} \end{cases}$$

(1) $x_1(t) = e^{j2mt}$ is a sinusoid with frequency $2m$, so the output is 0 in this case.

(2) $x_2(t) = \pi$ is a DC signal, so it passes through the LPF without distortion.

(3) By the table, the Fourier transform of $x_3(t) = \frac{2\pi}{m}\frac{\sin(mt/2)}{\pi t}$ is

$$X_3(\omega) = \begin{cases} 2\pi/m & \text{if } |\omega_0| \leq m/2 \\ 0 & \text{if } |\omega_0| > m/2. \end{cases}$$

Hence $X_3(\omega)H(\omega) = X_3(\omega)$, so there is no distortion.

(4) The Fourier transform of $x_4(t)$ is

$$X_4(\omega) = \int_{-\infty}^{\infty} x_4(t)e^{-j\omega t}\,dt = \int_0^b e^{-t(a+j\omega)}\,dt = \frac{1 - e^{-b(a+j\omega)}}{a + j\omega}$$

Hence $X_4(\omega)H(\omega) \neq X_4(\omega)$, so the received signal is distorted.