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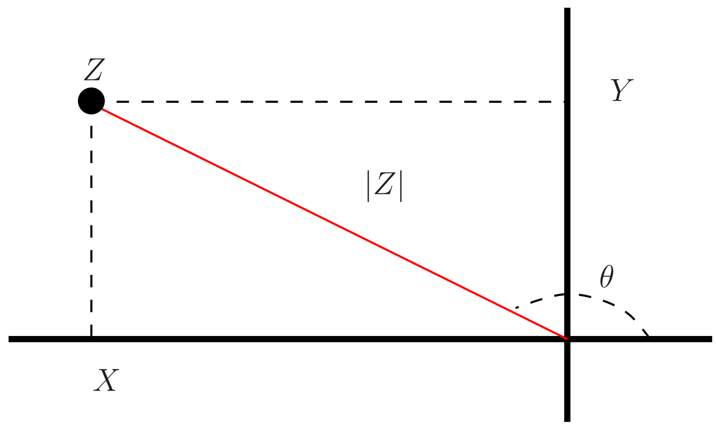
# ECE 45 Math Review Solutions




- **Why do we care about Complex Numbers?**

  While values such as voltage and current are always purely real numbers, it is often very useful to model time-dependent voltages and currents using mathematical transformations such as phasors, the Fourier Series, and the Fourier Transform (3 of 4 the main topics in the course).

  A strong understanding of complex numbers is critical to using and understanding these math tools that simplify circuit analysis.

- **Imaginary Numbers:**

  $j = \sqrt{-1} \;\Rightarrow\; \sqrt{-X} = j\,\sqrt{X}$

- **Complex Numbers:**

  Consist of *real* and *imaginary* parts, which can be represented in either *rectangular form* or *polar form.*

- **Rectangular Form:**

$$Z = X + jY$$

  $X$ and $Y$ are real numbers which are the *real* and *imaginary* components of $Z$.

- **Polar Form:**

$$Z = |Z|\, e^{j\theta}$$

  $|Z|$ and $\theta$ are real numbers which are the *magnitude* and *phase* components of $Z$, where $|Z| \geq 0$ and $0 \leq \theta < 2\pi$.

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- **Converting Between Representations:**

$$|Z| = \sqrt{X^2 + Y^2}, \quad \theta = \arctan(Y/X), \quad X = |Z|\,\cos(\theta), \quad Y = |Z|\,\sin(\theta).$$

These follow directly from trigonometry and viewing $Z$ as a point on a real and imaginary plot. Plotting a complex number is a useful for checking calculations.

- **Complex Conjugate:**

$$Z = X + jY = |Z|\,e^{j\theta} \iff Z^* = X - j\,Y = |Z|\,e^{-j\theta}$$
$$ZZ^* = \left(|Z|\,e^{j\theta}\right)\,\left(|Z|\,e^{-j\theta}\right) = |Z|^2$$

- **Adding/Subtracting Complex Numbers:**

$$Z_1 \pm Z_2 = (X_1 \pm X_2) + j\,(Y_1 \pm Y_2)$$

Adding/subtracting is much easier in rectangular form.

- **Multiplying Complex Numbers:**

$$Z_1 Z_2 = (X_1 X_2 - Y_1 Y_2) + j(X_1 Y_2 + X_2 Y_1) = |Z_1|\,|Z_2|\,e^{j(\theta_1 + \theta_2)}$$

- **Dividing Complex Numbers:**

$$Z_1/Z_2 = \frac{X_1 + jY_1}{X_2 + jY_2} = \frac{|Z_1|\,e^{j\,\theta_1}}{|Z_2|\,e^{j\,\theta_2}} = \frac{|Z_1|}{|Z_2|}\,e^{j\,(\theta_1 - \theta_2)}$$

Multiplying/dividing is much easier in polar form.

- **Euler's Formula:**

$$e^{j\theta} = \cos(\theta) + j\sin(\theta)$$

Follows directly from rectangular vs. polar representations. Can be used to show:

$$\cos(\theta) = \frac{e^{j\theta} + e^{-j\theta}}{2} \quad \text{and} \quad \sin(\theta) = \frac{e^{j\theta} - e^{-j\theta}}{2j}$$

which are useful for simplifying expressions and deriving trig identities.

**Problems**:

1. *Represent the following complex numbers in polar form:* $Z_1 = 1 - j$ *and* $Z_2 = -1 + j$

**Solutions**

$$Z_1 = \sqrt{2}\left(\tfrac{1}{\sqrt{2}} - j\,\tfrac{1}{\sqrt{2}}\right) \quad = \sqrt{2}\,e^{-j\pi/4}$$
$$Z_2 = \sqrt{2}\left(-\tfrac{1}{\sqrt{2}} + j\,\tfrac{1}{\sqrt{2}}\right) = \sqrt{2}\,e^{j3\pi/4}$$

We need to be careful of minus signs, since $\frac{-1}{1} = \frac{1}{-1}$, but $\theta_1 \neq \theta_2$. We can always plot a complex number in the complex plane to ensure our angles make sense.

2. *Represent the following complex number in rectangular form:* $\dfrac{e^{j2\pi/3}}{1 + j\,\sqrt{3}}$

**Solutions**

$$= \frac{e^{j2\pi/3}}{2\,(1/2 + j\,\sqrt{3}/2)} = \frac{e^{j2\pi/3}}{2\,e^{j\pi/3}} = \frac{1}{2}\,e^{j\pi/3} = \frac{1}{2}\left(\frac{1}{2} + j\,\frac{\sqrt{3}}{2}\right) = \frac{1}{4} + j\,\frac{\sqrt{3}}{4}$$

3. *Find the magnitude and phase of* $f(\omega) = \dfrac{1}{(j\omega + 1)^2}$

**Solutions**

$$j\omega + 1 = \sqrt{1 + \omega^2}\,e^{j\tan^{-1}\omega} \;\Rightarrow\; f(\omega) = \frac{1}{(1 + \omega^2)\,e^{2j\tan^{-1}\omega}} = \frac{e^{-2j\tan^{-1}\omega}}{1 + \omega^2}$$
$$\Rightarrow |f(\omega)| = \frac{1}{1 + \omega^2}, \quad \angle f(\omega) = -2\tan^{-1}\omega$$

4. *Simplify the following expression for a general integer* $n$*:* $(\cos(\pi n) + j\sin(\pi n))^n$

**Solutions**

$$\cos(\pi n) = (-1)^n, \text{ for all integer } n$$
$$\sin(\pi n) = 0, \text{ for all integer } n$$

$$\therefore (\cos(\pi n) + j\sin(\pi n))^n = ((-1)^n)^n = (-1)^{n^2} = (-1)^n.$$

**5.** *Find the magnitude and phase of $f(x)$ where $f(x) = 1$ when $x \geq 0$ and $f(x) = -j$ when $x < 0$.*

**Solutions**

$$|f(x)| = 1$$
$$\angle f(x) = \begin{cases} 0 & x \geq 0 \\ -\pi/2 & x < 0 \end{cases}$$

**6.** *Evaluate the following integral: $\int_0^\infty e^{-x} \cos(x)\, dx$.*

**Solutions**

Note that for any real numbers $a, b > 0$, we have

$$\lim_{x\to\infty} e^{-x(a+jb)} = \left(\lim_{x\to\infty} e^{-ax}\right)\left(\lim_{x\to\infty} e^{-jbx}\right) = 0\left(\lim_{x\to\infty} e^{-jbx}\right) = 0.$$

Using Euler's formula we have

$$\begin{aligned}
\int_0^\infty e^{-x} \cos(x)\, dx &= \int_0^\infty e^{-x} \frac{e^{jx} + e^{-jx}}{2}\, dx \\
&= \frac{1}{2}\int_0^\infty e^{-x(1-j)} + e^{-x(1+j)}\, dx \\
&= \frac{1}{2}\left(\frac{e^{-x(1-j)}}{-(1-j)} + \frac{e^{-x(1+j)}}{-(1+j)}\right)\Bigg|_0^\infty \\
&= \frac{-1}{2}\left(\frac{e^{-x(1-j)}(1+j) + e^{-x(1+j)}(1-j)}{(1-j)(1+j)}\right)\Bigg|_0^\infty \\
&= \frac{1}{2}\frac{(1+j)+(1-j)}{2} = \frac{1}{2}
\end{aligned}$$

Alternatively, we could use integration by parts and solve for $\int_0^\infty e^{-x} \cos(x)\, dx$.

**7.** *Find the derivative with respect to $x$ of $\frac{1}{j+x}$.*

**Solutions**

$$\frac{d}{dx}(j+x)^{-1} = -(j+x)^{-2} = -\frac{1}{(x+j)^2}$$

**8.** *If $f(x) = 1$ for $x$ between $-3$ and $2$ and is zero otherwise, for what values of $x$ does $f(4x) = 1$? For what values of $x$ does $f(x/2-3) = 1$?*

**Solutions**

$f(4x) = 1$ when $(4x)$ is between $-3$ and $2$. Or when $x$ is between $-3/4$ and $1/2$.

$f(x/2-3) = 1$ when $(x/2-3)$ is between $-3$ and $2$. Or when $x$ is between $0$ and $10$.

**9.** *Let $f(x) = \cos(x)$ when $x \geq 0$ and $f(x) = 0$ when $x < 0$. Let $g(x,y) = 1$ when $x < y$ and $g(x,y) = 0$ when $x \geq y$. Evaluate the following integral in terms of $y$: $\int_{-\infty}^{\infty} f(x)\, g(x,y)\, dx$.*

**Solutions**

Since $f(x) = 0$ when $x < 0$ and $f(x) = \cos(x)$ when $x \geq 0$, we have

$$\begin{aligned}\int_{-\infty}^{\infty} f(x)g(x,y)\, dx &= \int_{-\infty}^{0} 0\, g(x,y)\, dx + \int_{0}^{\infty} \cos(x)\, g(x,y)\, dx \\ &= \int_{0}^{\infty} \cos(x)\, g(x,y)\, dx\end{aligned}$$

$g(x,y) = 0$ when $x \geq y$, so if $y < 0$, then $\int_0^{\infty} \cos(x)\, g(x,y)\, dx = 0$, since $x \geq 0$.
If $y \geq 0$, we have

$$\int_0^{\infty} \cos(x)\, g(x,y)\, dx = \int_0^{y} \cos(x)\, dx + \int_y^{\infty} 0\, dx = \sin(y)$$

Thus

$$\int_{-\infty}^{\infty} f(x)\, g(x,y)\, dx = \begin{cases} \sin(y) & y \geq 0 \\ 0 & y < 0 \end{cases}$$

**10.** *For any positive real number $a$, evaluate $\int_0^{\infty} x\, e^{-(a+j)x}\, dx$*

**Solutions**

Recall integration by parts:

$$\int u dv = uv - \int v du$$

Let $u = x$ and $dv = e^{-(a+j)x}$. Then $du = dx$ and $v = -e^{-(a+j)x}/(a+j)$. So

$$\begin{aligned}\int_0^{\infty} x\, e^{-(a+j)x}\, dx &= x\, \frac{-e^{-(a+j)x}}{a+j}\Bigg|_0^{\infty} - \int_0^{\infty} \frac{-e^{-(a+j)x}}{a+j}\, dx \\ &= \frac{1}{a+j} \int_0^{\infty} e^{-(a+j)x}\, dx = \frac{-e^{-(a+j)x}}{(a+j)^2}\Bigg|_0^{\infty} = \frac{1}{(a+j)^2}\end{aligned}$$