University of California, San Diego
ECE 45 Spring 2018
Midterm Exam I

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Print your name: SOLUTIONS

Student ID Number: ______________________

Note: No books, notes, calculators, or other electronic devices allowed.

Question	Score
1	10 / 10
2	10 / 10
3	10 / 10
4	10 / 10
Total	40 / 40

Some Useful Formulas

Phasors: $$A\cos(\omega t+\phi) \overset{\omega}{\longleftrightarrow} Ae^{j\phi}$$

Impedances: Resistor: R Capacitor: $\dfrac{1}{j\omega C}$ Inductor: $j\omega L$

Euler's Formula:

$$e^{j\theta} = \cos(\theta) + j\sin(\theta)$$

$$\cos(\theta) = \frac{e^{j\theta} + e^{-j\theta}}{2}$$

$$\sin(\theta) = \frac{e^{j\theta} - e^{-j\theta}}{2j}$$

Fourier Series:

$$f(t) = \sum_{n=-\infty}^{\infty} F_n\, e^{j\omega_0 nt} \quad \text{where} \quad F_n = \frac{1}{T}\int_T f(t)\, e^{-j\omega_0 nt}\, dt$$

LTI System Transfer Function:

$$e^{j\omega_0 t} \longrightarrow \boxed{\text{LTI System}} \longrightarrow H(\omega_0)\, e^{j\omega_0 t}$$

$$A\cos(\omega_0 t+\phi) \longrightarrow \boxed{\text{LTI System}} \longrightarrow |H(\omega_0)|\, A\cos(\omega_0 t + \phi + \angle H(\omega_0))$$

Linear Systems:

$$x_1(t) \longrightarrow \boxed{\text{Linear System}} \longrightarrow y_1(t)$$

$$x_2(t) \longrightarrow \boxed{\text{Linear System}} \longrightarrow y_2(t)$$

implies

$$A_1x_1(t) + A_2x_2(t) \longrightarrow \boxed{\text{Linear System}} \longrightarrow A_1y_1(t) + A_2y_2(t)$$

Time-Invariant Systems:

$$x(t) \longrightarrow \boxed{\text{TI System}} \longrightarrow y(t)$$

implies

$$x(t-t_0) \longrightarrow \boxed{\text{TI System}} \longrightarrow y(t-t_0)$$

Problem 1: [10 points]

Suppose the signal $x(t)$ is periodic with period T and has Fourier series components

$$X_n = \begin{cases} T & \text{if } n = 0 \\ j/n & \text{if } n = -1, -2, \ldots \\ -j/n & \text{if } n = 1, 2, \ldots \end{cases}$$

$x(t)$ is the input to an LTI system with transfer function

$$H(\omega) = \begin{cases} j\omega & \text{if } \left|\omega - \frac{2\pi A}{T}\right| < \frac{2\pi}{T} \\ 0 & \text{otherwise.} \end{cases}$$

Write the output $y(t)$ as a purely real function.

V1: $T = 7$, $A = 2$ and **V2:** $T = 5$, $A = 3$

Solution: This problem is similar to **Problems 1.7, 2.5, 2.10**

By the LTI system property of the Fourier series, we know that

$$y(t) = \sum_{n=-\infty}^{\infty} X_n \, H\left(2\pi n/T\right) \, e^{j\frac{2\pi n}{T}t}.$$

We then have

$$H\left(\frac{2\pi n}{T}\right) = \begin{cases} j\frac{2\pi n}{T} & \text{if } \left|\frac{2\pi}{T}(n - A)\right| < \frac{2\pi}{T} \\ 0 & \text{otherwise.} \end{cases} = \begin{cases} j\frac{2\pi n}{T} & \text{if } n = A \\ 0 & \text{otherwise.} \end{cases}$$

and so

$$\begin{aligned} y(t) &= X_A \; H\left(2\pi A/T\right) \, e^{j\frac{2\pi A}{T}t} \\ &= \frac{2\pi}{T} e^{j\frac{2\pi A}{T}t} \end{aligned}$$

Note: There was a typo in the statement of this problem, so it is not possible to write $y(t)$ as a purely real function.

The intended transfer function was

$$H(\omega) = \begin{cases} j\omega & \text{if } \left|\omega \pm \frac{2\pi A}{T}\right| < \frac{2\pi}{T} \\ 0 & \text{otherwise.} \end{cases}$$

We then have

$$H\left(\frac{2\pi n}{T}\right) = \begin{cases} j\frac{2\pi n}{T} & \text{if } |\frac{2\pi}{T}(n \pm A)| < \frac{2\pi}{T} \\ 0 & \text{otherwise.} \end{cases} = \begin{cases} j\frac{2\pi n}{T} & \text{if } n = \pm A \\ 0 & \text{otherwise.} \end{cases}$$

and so

$$\begin{aligned} y(t) &= X_A \; H\left(2\pi A/T\right) \; e^{j\frac{2\pi A}{T}\,t} + X_{-A} \; H\left(-2\pi A/T\right) \; e^{-j\frac{2\pi A}{T}\,t} \\ &= \frac{2\pi}{T} \; \left(e^{j\frac{2\pi A}{T}\,t} + e^{-j\frac{2\pi A}{T}\,t}\right) \\ &= \frac{4\pi}{T} \; \cos\left(\frac{2\pi A}{T}\,t\right) \end{aligned}$$

Problem 2: [10 points]

Let $f(t) = |\cos(A\pi t)|$. Calculate the exponential Fourier series components F_n of $f(t)$. Simplify as much as possible.

Tip: You may find it helpful to sketch $f(t)$.

V1: $A = 5$ and **V2:** $A = 7$

Solution: This problem is similar to **Problems 2.7 – 2.10**.

$f(t)$ is periodic with period $1/A$, since

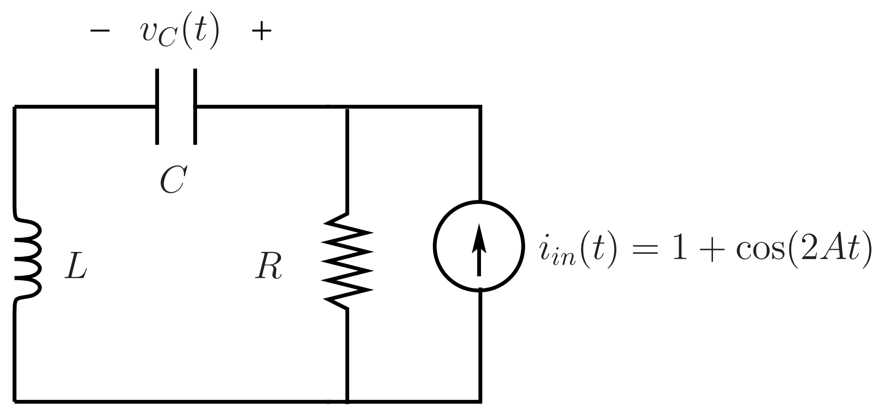
$$f(t + 1/A) = |\cos(A\pi t + \pi)| = |-\cos(A\pi t)| = |\cos(A\pi t)|.$$

The Fourier series components F_n of $f(t)$ are given by

$$\begin{aligned}
F_n &= A \int_{-1/(2A)}^{1/(2A)} \cos(A\pi t)\, e^{-j2A\pi n t}\, dt \\
&= \frac{A}{\pi} \int_{-1/(2A)}^{1/(2A)} \frac{e^{jA\pi t} + e^{-jA\pi t}}{2}\, e^{-j2A\pi n t}\, dt \\
&= \frac{A}{2} \int_{-1/(2A)}^{1/(2A)} e^{jA\pi\,(1-2n)\,t} + e^{-jA\pi\,(1+2n)\,t}\, dt \\
&= \frac{A}{2} \left(\frac{e^{jA\pi\,(1-2n)\,t}}{jA\pi\,(1-2n)} - \frac{e^{-jA\pi\,(1+2n)\,t}}{jA\pi\,(1+2n)} \right) \Bigg|_{-1/(2A)}^{1/(2A)} \\
&= \frac{1}{2j\pi} \left(\frac{e^{j\pi\,(1-2n)/2} - e^{-j\pi\,(1-2n)\,/2}}{1-2n} - \frac{e^{-j\pi\,(1+2n)/2} - e^{j\pi\,(1+2n)/2}}{1+2n} \right) \\
&= \frac{1}{2j\pi} \left(\frac{e^{j\pi/2} e^{-j\pi n} - e^{-j\pi/2} e^{j\pi n}}{1-2n} - \frac{e^{-j\pi/2} e^{-j\pi n} - e^{j\pi/2} e^{j\pi n}}{1+2n} \right) \\
&= \frac{1}{2j\pi} \left(\frac{(j)\,(-1)^n - (-j)\,(-1)^n}{1-2n} - \frac{(-j)\,(-1)^n - (j)\,(-1)^n}{1+2n} \right) \\
&= \frac{(-1)^n}{\pi} \left(\frac{1}{1-2n} + \frac{1}{1+2n} \right) = \frac{2\,(-1)^n}{\pi\,(1-4n^2)}
\end{aligned}$$

Problem 3: [10 points]

In the circuit below, determine the steady-state voltage $v_C(t)$.

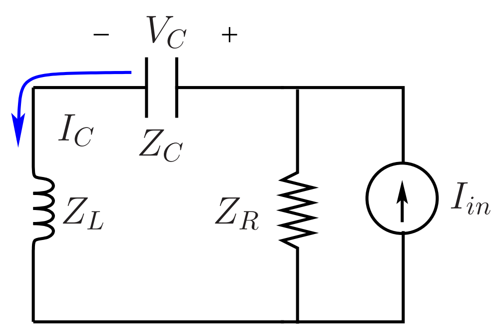


V1: $A = R = 2$, $C = 1$, $L = \frac{1}{4A^2} = \frac{1}{16}$

V2: $A = R = 3$, $C = 1$, $L = \frac{1}{4A^2} = \frac{1}{36}$

Solution: This problem is similar to **Problem 1.5**

Assume $i_{in}(t)$ is sinusoidal with phasor I_{in} and "phasorize" the circuit.



Then by using a current divider, we have

$$I_C = I_{in}\,\frac{1/(Z_C + Z_L)}{1/(Z_C + Z_L) + 1/Z_R} = I_{in}\,\frac{Z_R}{Z_R + Z_C + Z_L}$$

and so

$$V_C = I_C\,Z_C = I_{in}\,\frac{Z_R\,Z_C}{Z_R + Z_C + Z_L}$$

which implies

$$H(\omega) = \frac{V_C}{I_{in}} = \frac{R/(j\omega C)}{R + 1/(j\omega C) + j\omega L} = \frac{R}{(1 - \omega^2 LC) + j\omega RC}$$

We have $H(0) = R = A$, and

$$H(2A) = \frac{R}{(1 - 4A^2LC) + j2ARC} = \frac{1}{j2A}$$

so using the linearity of the system (i.e. superposition), we have

$$\begin{aligned} v_C(t) &= H(0) + |H(2A)| \cos(2At + \angle H(2A)) \\ &= A + \frac{1}{2A} \cos(2At - \pi/2) \\ &= A + \frac{1}{2A} \sin(2At) \end{aligned}$$

Problem 4: [10 points]

Sketch the magnitude (in dB) and phase Bode plots of the transfer function

$$H(\omega) = \frac{Mj\omega + (j\omega)^2}{100 - \omega^2 + 20j\omega}$$

Use the Bode plot to determine the output of this system when $\sin(Mt/5)$ is the input.

V1: $M = 3000$ and **V2:** $M = 4000$

Solution: This problem is similar to **Problems 2.2, 2.4**

We first write $H(\omega)$ in standard form:

$$H(\omega) = \frac{Mj\omega + (j\omega)^2}{100 - \omega^2 + 20j\omega} = \frac{j\omega\,(M + j\omega)}{(10 + j\omega)^2} = \frac{M\,j\omega\,(1 + j\omega/M)}{100\,(1 + j\omega/10)^2}$$

The critical points are

$$\begin{aligned}
A &: (\omega,\, \text{dB}) = \left(1,\ 20\log\frac{M}{100}\right) \\
B &: (\omega,\, \text{dB}) = \left(10,\ 20 + 20\log\frac{M}{100}\right) \\
C &: (\omega,\, \text{dB}) = (M,\, 0) \\
D &: (\omega,\, \text{deg}) = (1,\, 90^o) \\
E &: (\omega,\, \text{deg}) = (100,\, -90^o) \\
F &: (\omega,\, \text{deg}) = (M/10,\, -90^o) \\
G &: (\omega,\, \text{deg}) = (10M,\, 0^o)
\end{aligned}$$

Using the Bode plot, we have

$$\begin{aligned}
20\ \log|H(M/5)| &= 20 + 20\ \log\frac{M}{100} - 20\ \log\frac{M/5}{10} = 20\ \log 5 \\
\longrightarrow\ \ H(M/5) &= 5 \\
\angle H(M/5) &= -90^o + 45^o\log\frac{M/5}{M/10} = -90^o + 45^o\log 2
\end{aligned}$$

and so when $\sin(Mt/5)$ is the input, the output is approximately

$$5\ \sin(Mt/5 - 90^o + 45^o \log 2).$$