University of California, San Diego

ECE 45 Spring 2014

Final Examination SOLUTIONS

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Print your name: ____________________________________

Student ID Number: _________________________________

- No Books, No Notes, No calculators allowed

Question	Score
1	/35
2	/40
3	/55
4	/30
5	/40
Total:	/200

Problem 1) [35 pts]

Determine the minimum sampling frequency f_s in Hz needed to sample the following signals without aliasing:

a. [5 pts] $f(t) = \cos(6000\,\pi\, t) + 2$

b. [10 pts] $h(t) = f(t)\,\left(\frac{\sin(1000\,\pi\, t)}{\pi t}\right)$

c. [10 pts] $x(t) = \left(\frac{\sin(1000\,\pi\, t)}{\pi t}\right) * \left(\frac{\sin(1000\,\pi\, t)}{\pi t}\right)$

d. [10 pts] Determine the energy of the signal $h(t)$ i.e. $\int_{-\infty}^{\infty} |h(t)|^2 dt$

a.

$$F(j\omega) = \pi[\delta(\omega - 6000\,\pi) + \ \delta(\omega + 6000\,\pi)] + \ 2\pi\ \delta(\omega)$$

$$\rightarrow \omega_{max} = 6000\,\pi \rightarrow f_s > \frac{2\ \omega_{max}}{2\ \pi} = \ \boldsymbol{6kHz}$$

b.

$$G(j\omega) = \begin{cases} 1 & |\omega| < 1000\pi \\ 0 & |\omega| > 1000\pi \end{cases}$$

$$H(j\omega) = \frac{1}{2\pi} F(j\omega) * G(j\omega) = \frac{1}{2}\left[G\big(j(\omega - 6000\pi)\big) + \ G\big(j(\omega + 6000\pi)\big)\right] + G(j\omega)\big)$$

$$\rightarrow \omega_{max} \ = 7000\,\pi \rightarrow f_s > \frac{2\ \omega_{max}}{2\ \pi} = \ \boldsymbol{7kHz}$$

c.

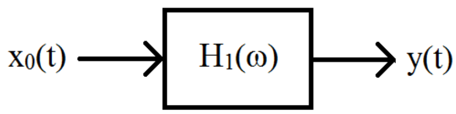
$$X(j\omega) = G(j\omega)G(j\omega) = G(j\omega) = \ \begin{cases} 1 & |\omega| < 1000\pi \\ 0 & |\omega| > 1000\pi \end{cases} \rightarrow \omega_{max} = 1000\pi$$

$$\rightarrow f_s > \frac{2\ \omega_{max}}{2\ \pi} = \ \boldsymbol{1kHz}$$

d.

$$\int_{-\infty}^{\infty} |h(t)|^2 dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} |H(j\omega)|^2 d\omega$$
$$= \frac{1}{2\pi}\int_{-7000\pi}^{-5000\pi} \left(\frac{1}{2}\right)^2 d\omega + \frac{1}{2\pi}\int_{5000\pi}^{7000\pi} \left(\frac{1}{2}\right)^2 d\omega + \frac{1}{2\pi}\int_{-1000\pi}^{1000\pi} 1\, d\omega$$
$$= \frac{1}{2\pi}\left[\frac{1}{4} 2000\,\pi + \frac{1}{4} 2000\pi + 2000\pi\right] = \mathbf{1500}$$

Problem 2) [40 pts] Suppose you are trying to play your favorite song for your friend who lives very far away. Unfortunately, your friend does not have a computer, so we are forced to send the analog, continuous time signal of the song, $x_0(t)$. We are able to model the connection to your friend as the LTI system $H_1(\omega)$. In order for your friend to be able to correctly hear your song, we must have $y(t) = x_0(t)$ for the system below.

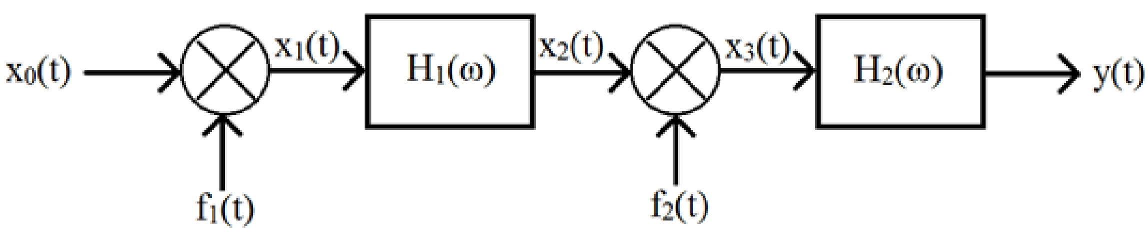


where $FT\big(x_0(t)\big) = X_0(j\omega) \neq 0$ when $|\omega| < 2\pi\ 10^4$ and $X_0(j\omega) = 0$ when $|\omega| > 2\pi\ 10^4$

and $H_1(\omega) = \begin{cases} 1 & 4\,\pi\ 10^4 \leq \omega \leq \pi\ 10^5 \\ 0 & else \end{cases}$

a. **[10 pts]** Show that with this set up, your friend will not correctly hear your song.

b. **[30 pts]** Luckily, you have taken ECE45 and propose that, with some modifications to our system, your friend can hear your song. You claim that we can select $f_1(t)$, $f_2(t)$, and $H_2(\omega)$ such that $y(t) = x_0(t)$. What should these functions be? Show that your friend can correctly hear your song.
(Hint: consider using modulation/frequency shifting)



Note: $x_1(t) = x_0(t)\ f_1(t)$ and $x_3(t) = x_2(t)\ f_2(t)$

a. $H_1(\omega) = 0$ whenever $X_0(j\omega) \neq 0$, so $Y(j\omega) = X_0(j\omega)H_1(\omega) = 0 \rightarrow \boldsymbol{y(t) = 0 \neq x_0(t)}$

b. $\boldsymbol{f_1(t) = f_2(t) = \cos(7\pi\, 10^4 t)}$, $\boldsymbol{H_2(\omega) = \begin{cases} 2 & |\omega| \leq 3\pi 10^4 \\ 0 & |\omega| > 3\pi 10^4 \end{cases}}$

$$F_1(j\omega) = F_2(j\omega) = \pi[\delta(\omega + 7\pi 10^4) + \delta(\omega - 7\pi 10^4)]$$

$$x_1(t) = x_0(t)\, f_1(t) \leftrightarrow X_1(j\omega) = \frac{1}{2\pi} X_0(j\omega) * F_1(j\omega)$$

$$= \frac{1}{2}\left[X_0\big(j(\omega - 7\,\pi\, 10^4)\big) + X_0\big(j(\omega + 7\,\pi\, 10^4)\big)\right]$$

$$X_2(j\omega) = H_1(\omega)X_1(j\omega) = \frac{1}{2} X_0\big(j(\omega - 7\,\pi\, 10^4)\big)$$

$$x_2(t) = x_2(t)\, f_2(t) \leftrightarrow X_3(j\omega) = \frac{1}{2\pi} X_2(j\omega) * F_2(j\omega)$$

$$= \frac{1}{2}\left[X_1\big(j(\omega - 7\,\pi\, 10^4)\big) + X_1\big(j(\omega + 7\,\pi\, 10^4)\big)\right]$$

$$= \frac{1}{4}\left[2\, X_0(j\omega) + X_0\big(j(\omega - 7\,\pi\, 10^4)\big) + X_0\big(j(\omega + 7\,\pi\, 10^4)\big)\right]$$

$$\rightarrow Y(j\omega) = H_2(\omega)X_3(j\omega) = X_0(j\omega)$$

$$\rightarrow \boldsymbol{y(t) = x_0(t)}$$

Problem 3) [55 pts]

a. **[35 pts]** Compute and sketch $x(t) = f(t) * g(t)$ where

$$g(t) = \begin{cases} t & -1 \le t \le 1 \\ 0 & else \end{cases} \quad \text{and} \quad f(t) = \begin{cases} 1 & 0 \le t < 2 \\ 0 & else \end{cases}$$

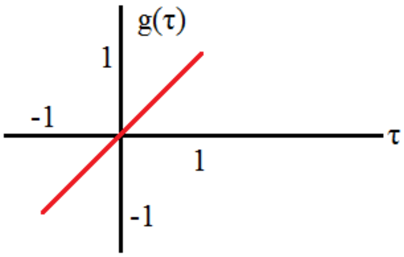
b. **[15 pts]** Determine $y(t) = f(t) * h(t)$ where

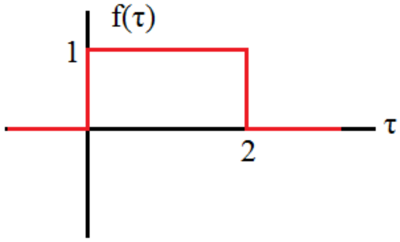
$$h(t) = \begin{cases} t-1 & 0 \le t \le 2 \\ 3-t & 2 < t \le 4 \\ 0 & else \end{cases}$$

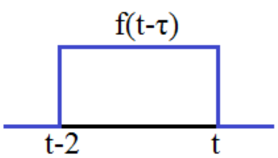
(Hint: this can be done without any additional computation. Try to write $h(t)$ in terms of $g(t)$ and use the result from part a.)

a.

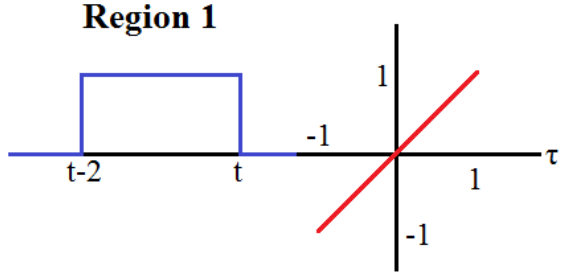
$$x(t) = f(t) * g(t) = \int_{-\infty}^{\infty} g(\tau) f(t-\tau)\, d\tau$$



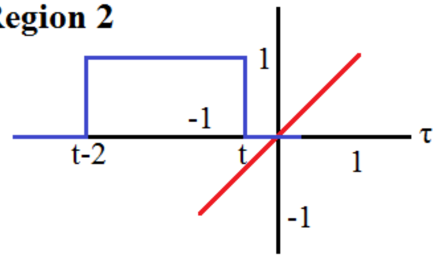




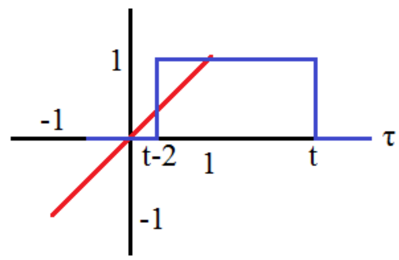
Region 1



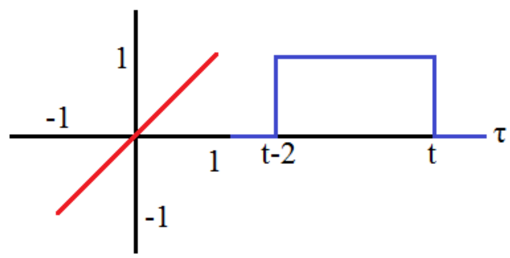
Region 2



Region 3



Region 4



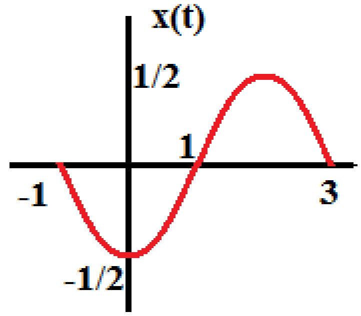
Region 1: $t < -1$ No overlap

Region 2: $-1 \leq t < 1$ Overlap from $\tau = -1$ to t

Region 3: $1 \leq t < 3$ Overlap from $\tau = t - 2$ to 1

Region 4: $t \geq 3$ No overlap

$$x(t) = \begin{cases} \int_{-1}^{t} \tau \, d\tau & -1 \le t < 1 \\ \int_{t-2}^{1} \tau \, d\tau & 1 \le t < 3 \\ 0 & else \end{cases} = \begin{cases} \mathbf{\frac{1}{2}(t^2 - 1)} & \mathbf{-1 \le t} < 1 \\ \mathbf{\frac{1}{2}(1 - (t-2)^2)} & \mathbf{1 \le t} < 3 \\ \mathbf{0} & \mathbf{else} \end{cases}$$



b.

Note: $h(t) = g(t-1) - g(t-3)$

so $y(t) = f(t) * g(t-1) - f(t) * g(t-3) = \boldsymbol{x(t-1) - x(t-3)}$

Problem 4) [30 pts]

a. **[20 pts]** Write $y(t)$ in its simplest form, where $y(t) = x(t) * (u(t+1) - u(t-1))$ and $FT\big(x(t)\big) = X(j\omega) = \pi\,[\delta(\omega - 2\pi) + \delta(\omega - 1) + \delta(\omega + 1)]$

b. **[5 pts]** Is $x(t)$ periodic? If so, what is its period?

c. **[5 pts]** Is $y(t)$ periodic? If so, what is its period?

a.

let $f(t) = \big(u(t+1) - u(t-1)\big) = \begin{cases} 1 & -1 \le t < 1 \\ 0 & else \end{cases} \leftrightarrow F(j\omega) = \frac{2\sin(\omega)}{\omega}$

$$Y(j\omega) = X(j\omega)F(j\omega) = \pi[\delta(\omega - 2\pi) + \delta(\omega - 1) + \delta(\omega + 1)]\,\frac{2\sin(\omega)}{\omega}$$

$$= \pi\left[\delta(\omega - 2\pi)\,e^{-j2\pi}\,\frac{2\sin(2\pi)}{2\pi} + \delta(\omega - 1)\frac{2\sin(1)}{1} + \delta(\omega + 1)\frac{2\sin(-1)}{-1}\right]$$

$$= \pi\delta(\omega - 1)\,2\sin(1) - \pi\delta(\omega + 1)\,2\sin(-1)$$

$$= 2\pi\sin(1)\,[\delta(\omega - 1) + \delta(\omega + 1)] \leftrightarrow y(t) = \mathbf{2\sin(1)\ \cos(}\boldsymbol{t}\mathbf{)}$$

b.

$$X(j\omega) = \pi\,[\delta(\omega - 2\pi) + \delta(\omega - 1) + \delta(\omega + 1)] \to x(t) = \frac{e^{j2\pi t}}{2} + \cos(t)$$

$$= \frac{(\cos(2\pi t) + j\sin(2\pi t) + 4\cos(t))}{4}$$

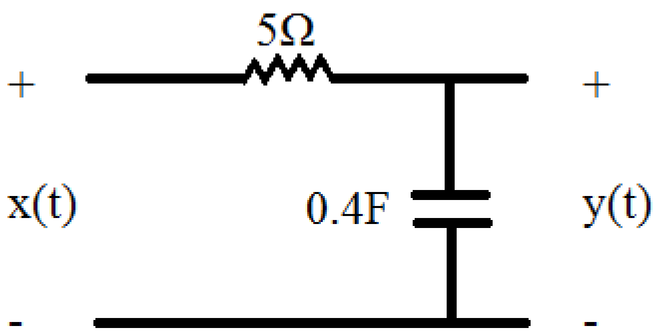
x(t) is not period. $\cos(2\pi t)$ and $j\ \sin(2\pi t)$ are periodic with period $T_1 = 1$, and $4\cos(t)$ is periodic with period $T_1 = 2\pi$, but there does not exist a T such that T is an integer multiple of both T_1 and T_2

c.

$y(t) = 2\sin(1)\cos(t)$

y(t) is periodic with period 2$\boldsymbol{\pi}$

Problem 5) [45 pts]



a. **[10 pts]** Determine the impulse response $h(t)$ of the above circuit

b. **[15 pts]** What is $y(t)$ when the input to is $x(t) = e^{-2t}\,u(t)$

c. **[20 pts]** If the output of a (different) system is $y(t) = e^{-2(t-1)}u(t-1)$ when the input is $x(t) = u(t)$, find the transfer function $H(\omega)$

a.

$$Y(j\omega) = X(j\omega)\frac{\frac{1}{j\omega C}}{R + \frac{1}{j\omega C}} = X(j\omega)\frac{1/RC}{j\omega + 1/RC}$$

$$H(\omega) = \frac{Y(j\omega)}{X(j\omega)} = \frac{1}{RC}\,\frac{1}{j\omega + 1/RC} \;\rightarrow\; h(t) = \frac{1}{RC}\,e^{-\frac{t}{RC}} = \mathbf{\frac{1}{2}\,e^{-\frac{1}{2}t}}$$

b.

$$X(j\omega) = \frac{1}{2 + j\omega}$$

$$Y(j\omega) = X(j\omega)H(\omega) = \frac{1/2}{(2 + j\omega)(1/2 + j\omega)} = \frac{A}{2 + j\omega} + \frac{B}{1/2 + j\omega}$$

$$\rightarrow \tfrac{1}{2} = A\left(\tfrac{1}{2} + j\omega\right) + B(2 + j\omega) \;\rightarrow\; \begin{matrix} 1/2 = A/2 + 2B \\ 0 = A + B \end{matrix}$$

$$Y(j\omega) = -\frac{1/3}{2 + j\omega} + \frac{1/3}{1/2 + j\omega} \;\rightarrow\; \boldsymbol{y(t) = \frac{1}{3}\left(e^{-\frac{1}{2t}} - e^{-2t}\right)}$$

c.

$$Y(j\omega) = \frac{e^{-j\omega}}{2 + j\omega}$$

$$y(t) = u(t) * h(t) \rightarrow \frac{dy(t)}{dt} = \frac{du(t)}{dt} * h(t) = \delta(t) * h(t) = h(t)$$

$$h(t) = \frac{dy(t)}{dt} \rightarrow H(j\omega) = j\omega Y(j\omega) = \boldsymbol{e^{-j\omega}\frac{j\omega}{2 + j\omega}}$$