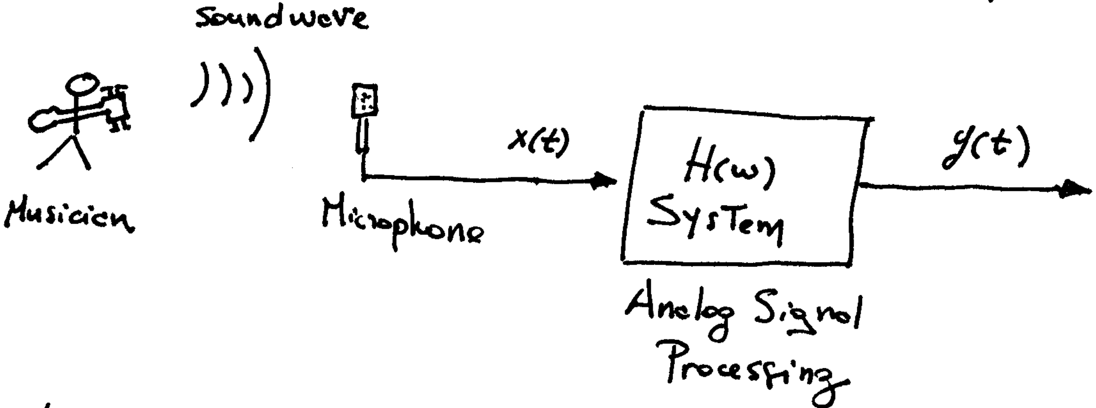
ECE 45
Prof. Franceschetti
LECTURE NOTES #10

SAMPLING THEOREM

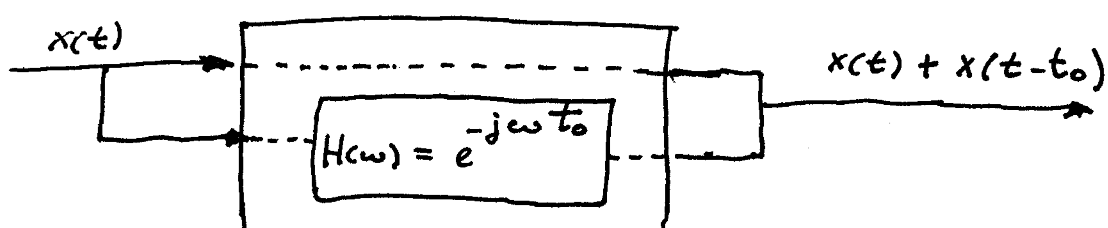
We want to start introducing one of the greatest achievements in engineering and signals and systems: the sampling theorem.
To do this we have to do some more math of Fourier Transforms.
But let's start with a practical example.



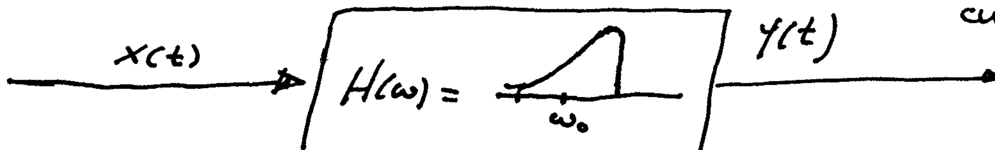
This is the chain we are familiar with up to now. We have an analog (continuous) signal that we are able to process using a System.

EXAMPLE

- Add an echo (delay) to the sound



- Bass / Treble control (filter)

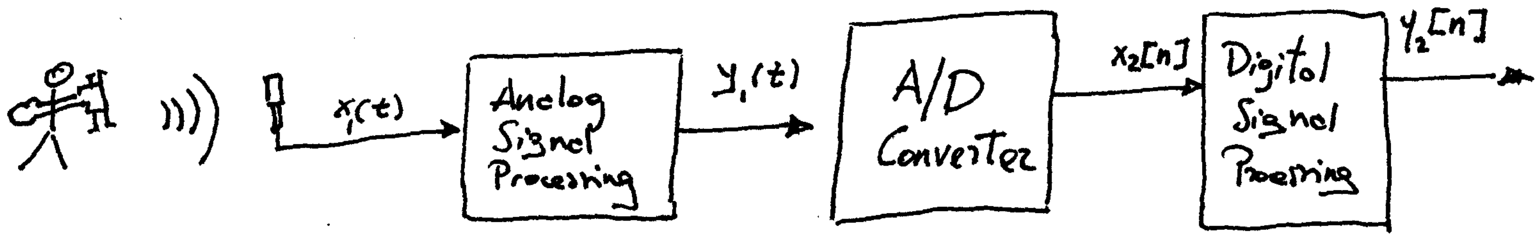


emphasize a certain range of frequencies
cut another range of frequencies

For example, if we want to emphasize the guitar sound we emphasize the high-frequencies. The vocals will be the mid-range frequencies, the bass & drums the low frequencies. We can design $H(\omega)$ appropriately to do this.

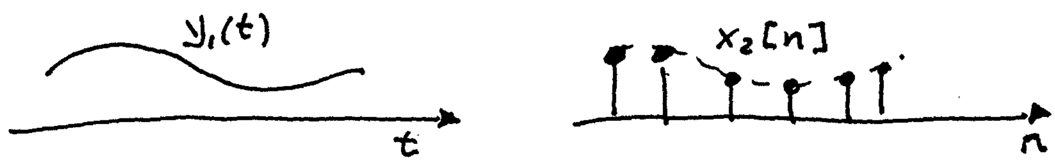
Now, what sampling allows us to do is to convert from the analog (continuous world) to the Digital (discrete world). This is very useful for two reasons:

(1) The processing can be done in Software
(2) Better sound (signal) quality, no noise.



EXAMPLE

- An example of A/D converter. Suppose the musician wants to record his song on a CD. He or she plugs the microphone in the computer where there is some hardware to perform the conversion to digital. At this point the music signal becomes a discrete sequence $x_2[n]$ which can be represented in bits.



Computer software can now digitally process this sequence. Furthermore it can store it on the Hard Disc or "burn" a cd.

To play back the music the inverse operations are performed: read the sequence $x[n]$ from the disc, perform conversion from digital to analog, send analog signal to the speakers, enjoy the music.

The main step in all of this is the conversion from analog to digital and vice-versa. To understand this step we have to learn the Sampling Theorem.

So far, we have seen some special FT's

$$1 \xleftrightarrow{FT} 2\pi\,\delta(\omega)$$

$$u(t) \xleftrightarrow{FT} \pi\delta(\omega) + \frac{1}{j\omega}$$

Let's see some more that we need.

Let $\quad X(j\omega) = 2\pi\,\delta(\omega-\omega_0)$

$$x(t) = \frac{\cancel{2\pi}}{\cancel{2\pi}} \int_{-\infty}^{+\infty} \delta(\omega-\omega_0)\, e^{j\omega t}\, dt = e^{j\omega_0 t}$$

So the F.T. of a complex expo at frequency ω_0 is a δ-function centered at ω_0 times a factor 2π.

$$e^{j\omega_0 t} \xleftrightarrow{FT} 2\pi\,\delta(\omega-\omega_0)$$

From this we can easily find the transform of sine and cosine using EULER'S formula. Do IT as an EXERCISE!

$$\cos(\omega_0 t) \xleftrightarrow{FT} \pi\,\delta(\omega-\omega_0) + \pi\,\delta(\omega+\omega_0)$$

$$\sin(\omega_0 t) \xleftrightarrow{FT} \frac{\pi}{j}\,\delta(\omega-\omega_0) - \frac{\pi}{j}\,\delta(\omega+\omega_0)$$

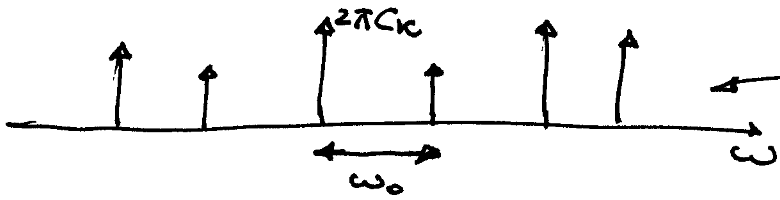
Taking one step further, we can compute the FT of a periodic signal $x(t)$.
Let's first write $x(t)$ as a Fourier Series

$$\boxed{x(t) = \sum_{k=-\infty}^{+\infty} c_k e^{jk\omega_0 t}}$$

By applying superposition, we can transform term by term:

$$\boxed{X(j\omega) = \sum_{k=-\infty}^{+\infty} c_k \, 2\pi \, \delta(\omega - k\omega_0)}$$

So we say that the FT of a periodic signal $x(t)$ is a "train of pulses", each pulse weighted by the corresponding Fourier coeff. c_k and multiplied by 2π.

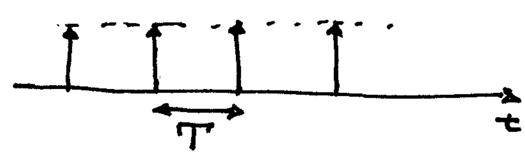


Transform of a periodic signal $x(t)$

The spacing between pulses is ω_0.

Now consider the signal

$$x(t) = \sum_{k=-\infty}^{+\infty} \delta(t - kT)$$



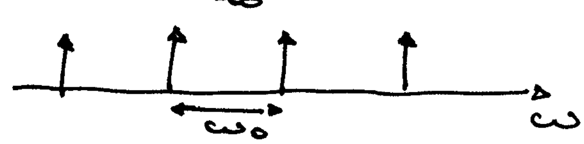
The Fourier Series of this is:

$$x(t) = \frac{1}{T} \sum_{k=-\infty}^{+\infty} e^{jk\omega_0 t}$$

since $c_k = \frac{1}{T} \int_{-T/2}^{T/2} \delta(t) e^{-j\omega_0 t} dt = \frac{1}{T}$

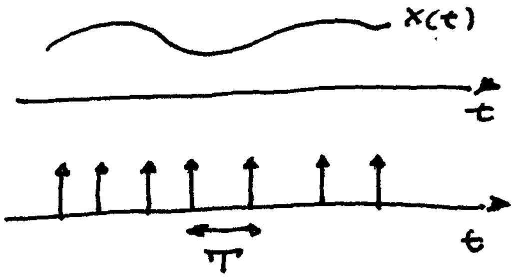
From which it follows

$$X(j\omega) = \frac{2\pi}{T} \sum_{k=-\infty}^{+\infty} \delta(\omega - k\omega_0) = \omega_0 \sum_{-\infty}^{+\infty} \delta(\omega - k\omega_0)$$

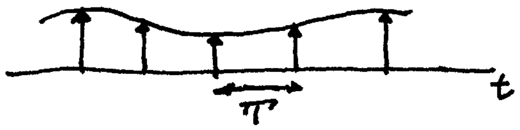


So we have that the FT of a train of pulses is still a train of pulses in the frequency domain, scaled by the factor ω_0.

We are almost ready for the sampling theorem ...
Let's consider a signal $x(t)$ multiplied by a train of δ-functions:



$$x(t) \cdot \sum_{k=-\infty}^{+\infty} \delta(t-kT) = \sum_{k=-\infty}^{+\infty} x(t)\,\delta(t-kT)$$



Let's compute the FT of this product:

$$FT\left(x(t) \cdot \sum_{k=-\infty}^{+\infty} \delta(t-kT)\right) = \frac{\omega_0}{2\pi} X(j\omega) \circledast \sum_{k=-\infty}^{+\infty} \delta(\omega-k\omega_0)$$

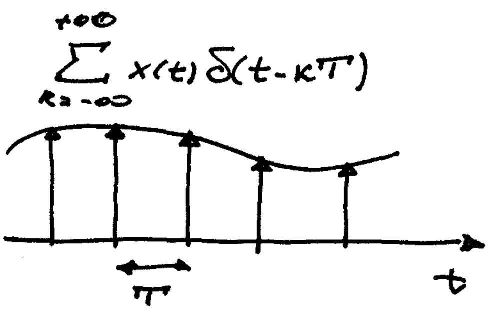
$$= \frac{1}{T} \sum_{k=-\infty}^{+\infty} X(j\omega) \circledast \delta(\omega-k\omega_0)$$

$$= \frac{1}{T} \sum_{k=-\infty}^{+\infty} X(j(\omega-k\omega_0))$$

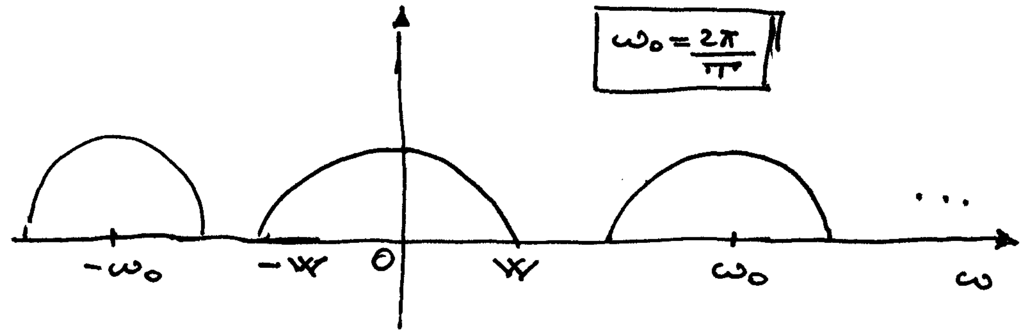
where the last equality follows from the convolution property of the δ-function (See lecture #8) $\int_{-\infty}^{+\infty} x(\tau)\,\delta(t-\tau)\,d\tau = x(t)$

$$X(j\omega) \circledast \delta(\omega-k\omega_0) = \int X(j\omega')\,\delta(\omega-k\omega_0-\omega')\,d\omega'$$
$$= X(j(\omega-k\omega_0)).$$

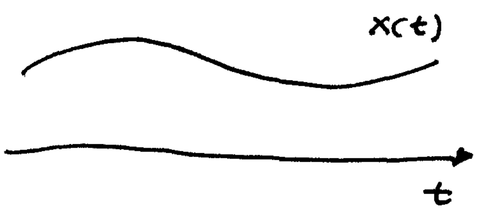
So we see that multiplication by a train of δ-s corresponds to "replicate" the signal in frequency domain:



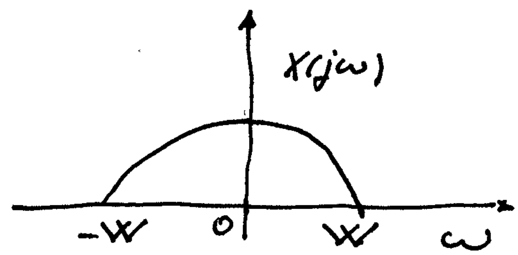
Sampled signal



Replicated Spectrum



original signal $x(t)$



Spectrum of $x(t)$

Notice that the sampled signal is a train of δ's each weighted by a coefficient that corresponds to $x(t)$ evaluated at $t = kT'$. These coefficients represent the entire signal.

In fact, by applying a low-pass filter to a train of δ's weighted by these coefficients we can reconstruct $x(t)$ provided that:

1. $x(t)$ has finite bandwidth $2W < \infty$
2. $\omega_0 > 2W$ sampling frequency is more than twice bandwidth.

MORE ON THIS in NEXT LECTURE