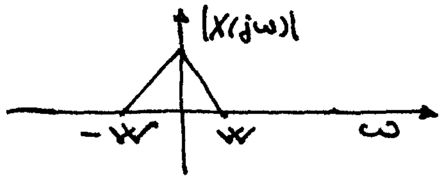
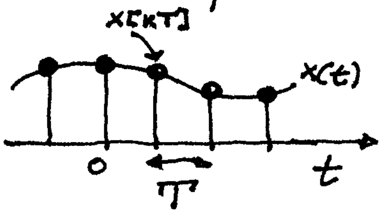
ECE 45
Prof. Franceschetti
Lecture notes # 11

In the last lecture we have introduced the SAMPLING THEOREM

Let $x(t)$ be a band-limited signal which means $|X(j\omega)| = 0$ for $|\omega| > W$



If samples of $x(t)$ are taken sufficiently close together in relation to the highest frequency of the signal, then these discrete samples completely specify the entire signal



samples $X[KT]$ $K = 0, \pm 1, \pm 2$

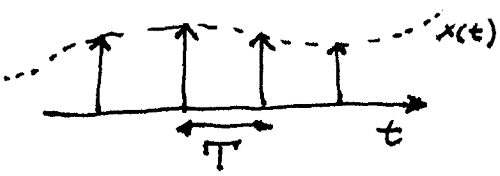
$\boxed{\omega_0 = \frac{2\pi}{T}}$ SAMPLING FREQUENCY

$\boxed{\omega_0 > 2W}$

How can we reconstruct the signal $x(t)$ from the discrete samples $x[K]$?

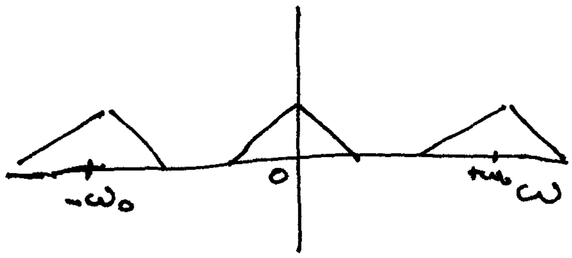
① Construct an impulse train with each δ-function weighted by the coefficient $x[K]$ ~~and then apply a low-pass filter~~

$$\sum_{K=-\infty}^{+\infty} x(t)\,\delta(t-KT)$$

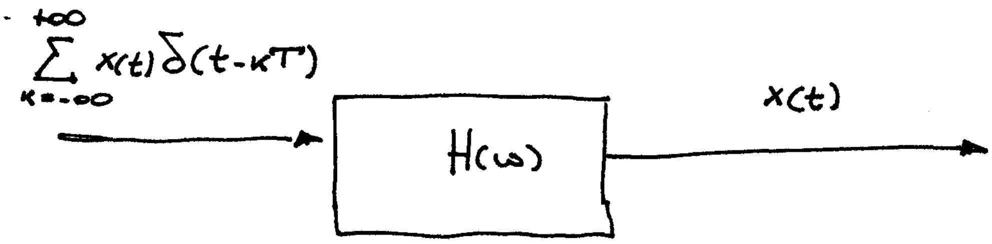


FT

This in the Fourier domain corresponds to constructing

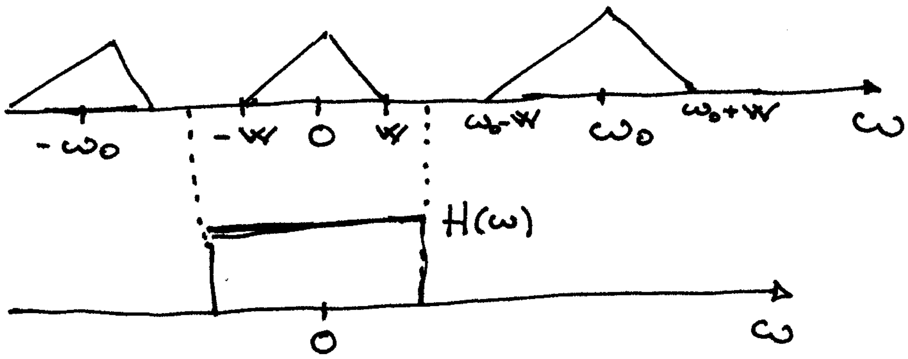


② Now apply a low-pass filter to the obtained signal

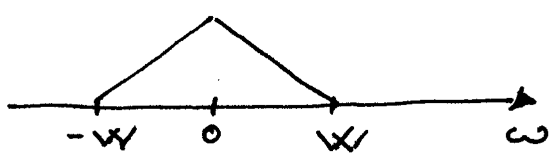


$\omega_0 - W > W$

$\boxed{\omega_0 > 2W}$



Multiplying the two spectra above we obtain:



which is the spectrum of $x(t)$ as desired.

Notice that the condition $\omega_0 > 2W$ is needed to ensure perfect reconstruction of $x(t)$. If $\omega_0 < 2W$ the replices would overlap and we would have some distorsion in the reconstructed signal.

For exemple, our ears are sensible to sounds up to 20KHz
So to capture music digitally and being able to reproduce it without distorsion what is the sampling rate we need?

$W = 20\,KHz$

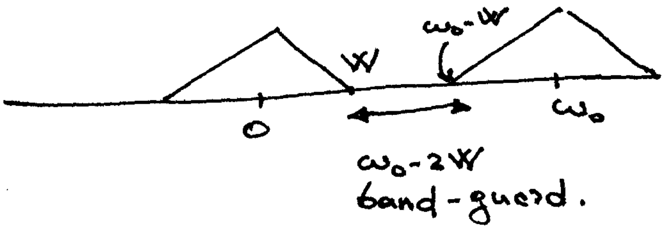
$\omega_0 > 2W = 40KHz$

$T < \frac{2\pi}{40KHz}$

In fact, digital music is typically sempled at 44 KHz

QUESTION : why 44 KHz and not exactly 40 KHz

The extra 4 KHz provide some "slack" commonly referred to as guard-band. To ensure separation of the replicas and simplify the design of the low-pass filter.

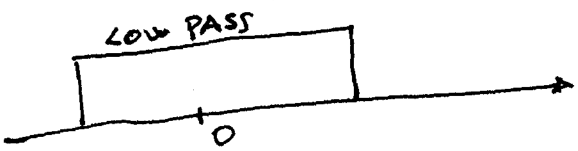


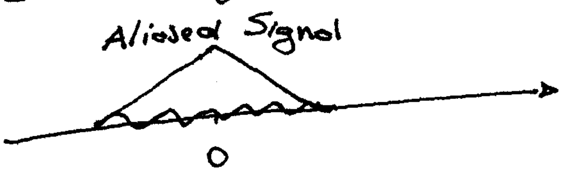
So is digital a "perfect" Technology ?

SOME PRACTICAL ISSUES

There are some other implementation issues though.
If the signal is not fully bandlimited, even if our ear is not sensible to the very high frequencies, these will generate some noise:

this signal is only approximately bandlimited to W when we replicate the spectrum, we obtain:





so when we low-pass this signal we still have some spurios high frequency components of the original signal that have been transferred in the low frequency regime and that we will hear as noise:

This kind of distorsion is called ALIASING

Solution: apply first a filter to the original signal to ensure it is band-limited.

Similar practical problems arise because

1. Filters are not ideal cut-offs in practice
2. We cannot generate an ideal train of pulses in practice.

For these and other reasons digital technology continues to co-exist with analog technology. In the music-reproduction business some for example claim that all the manipulations and processing required for digital reproduction (filtering etc.) eventually lead to music that sounds clean but "reconstructed" and not natural.

However, due to lower costs, and easier software processing tools, and indeed the ability to produce a very clean sound, digital technology has seen a tremendous growth in the last decades:

CD's – DVD – Hard disc recording – MP3's – Digital Sound effects – IPOD ...

MODULATION

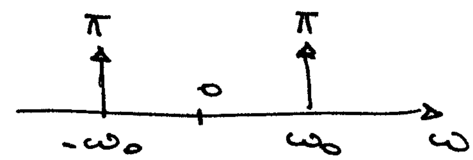
What we have learned regarding the sampling Theorem is also very useful to understand signal modulation.

Recall the F.T. of the complex exponential:

$$e^{j\omega_0 t} \xleftrightarrow{FT} 2\pi\,\delta(\omega-\omega_0)$$

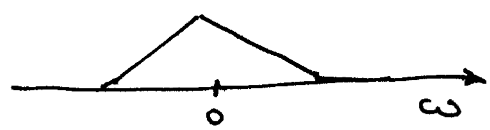
Let $x(t) = \cos\omega_0 t = \frac{1}{2}e^{j\omega_0 t} + \frac{1}{2}e^{-j\omega_0 t}$

$$FT(\cos\omega_0 t) = \pi\,\delta(\omega-\omega_0) + \pi\,\delta(\omega+\omega_0)$$



Now let's consider a generic signal $x(t)$

of spectrum $X(j\omega)$



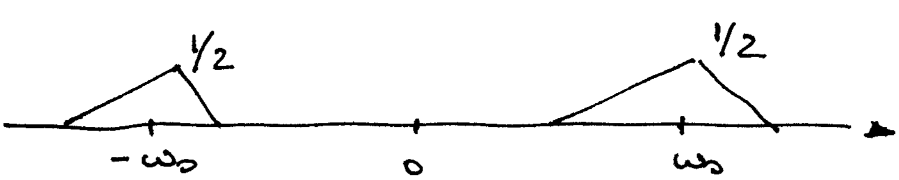
And let's multiply this by $\cos(\omega_0 t)$

$$x(t)\cos(\omega_0 t) \xleftrightarrow{FT} X(j\omega) \circledast \pi\,\frac{1}{2\pi}\delta(\omega-\omega_0)$$

$$+\, X(j\omega) \circledast \pi\,\frac{1}{2\pi}\delta(\omega+\omega_0)$$

$$= \frac{1}{2}X(j(\omega-\omega_0)) + \frac{1}{2}X(j(\omega+\omega_0))$$

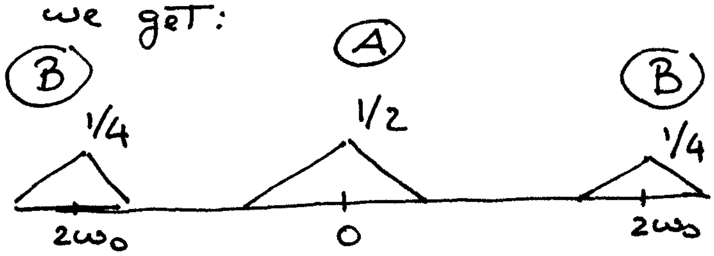
So the effect is that of creating two replicas of the spectrum of $x(t)$ and to shift them in frequency, placing them at $\omega = \pm\omega_0$ each scaled by a factor $1/2$



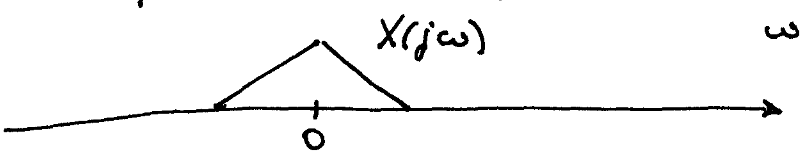
This operation is called modulation

To demodulate the signal we multiply again by a $\cos(\omega_0 t)$ and then apply a low-pass filter of gain 2.

By multiplying in time ~~sp~~ domain again by $\cos(\omega_0 t)$ we get:



And after a low pass filter and multiplication by a factor 2:



we obtain ~~back~~ the original signal.

Analytically, we have:

$$x(t)\cos(\omega_0 t)\cos(\omega_0 t) = x(t)\cos^2(\omega_0 t) =$$

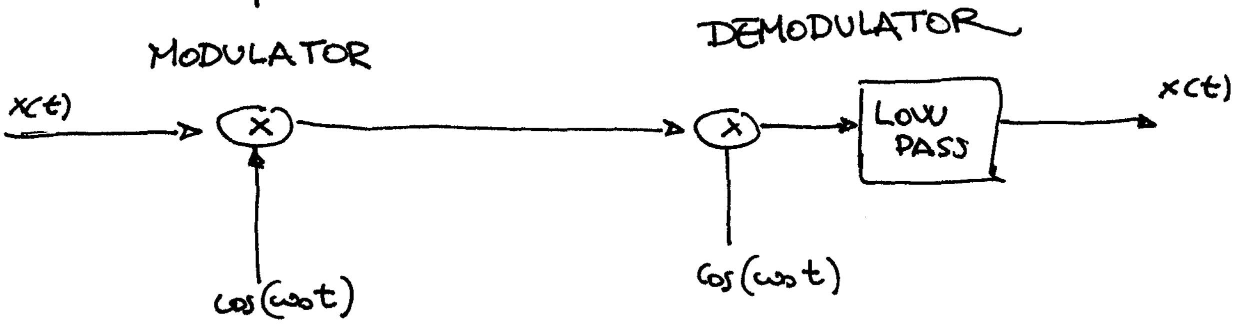
$$= x(t)\left[\frac{1}{2} + \frac{1}{2}\cos(2\omega_0 t)\right]$$

$$= \frac{1}{2}x(t) + \frac{1}{2}x(t)\cos(2\omega_0 t)$$

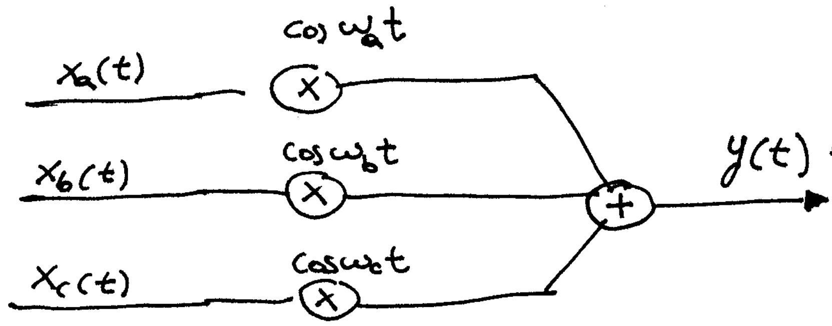
this is the central component A of the spectrum

these are the lateral components B of the spectrum centered at $\pm\omega_0$

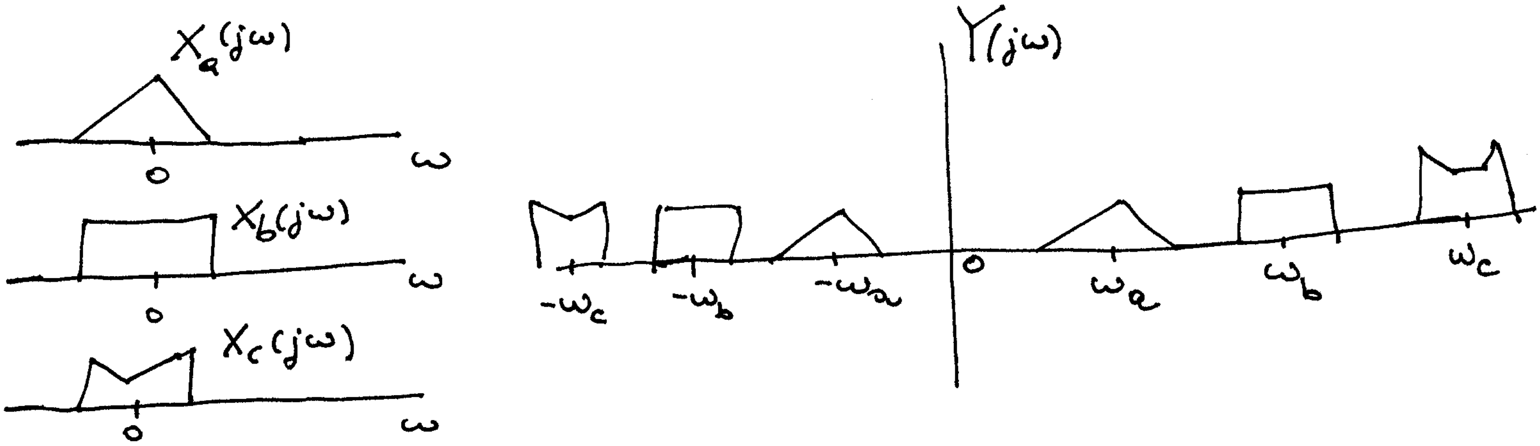
So, in summary we have



This is used to do Frequency Multiplexing of different signals:



$$y(t) = x_a(t)\cos(\omega_a t) + x_b(t)\cos(\omega_b t) + x_c(t)\cos(\omega_c t)$$



Suppose you have 3 radio stations each broadcasting their own signal, the signals are first modulated an placed in an appropriate frequency band and then broadcasted.

In this way the receiver can listen to the selected station by appropriate demodulation of the signal of interest, bringing it back to base-band and applying a low-pass filter.