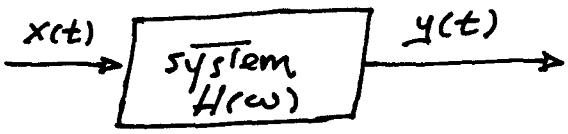
ECE 45
Prof. Franceschetti
Lecture Notes # 4

# FOURIER SERIES

So far we have seen the phasor method and the frequency response to find the response of a linear system to a sinusoidal signal at <u>any</u> frequency $\omega$.

We also know by now that for a <u>linear</u> system the principle of superposition holds and we can find the response to a linear combination of sinusoidal signals. For example




$$x(t) = a_1 \sin(\omega_1 t + \phi_1) + a_2 \sin(\omega_2 t + \phi_2)$$

$$y(t) = a_1 |H(\omega_1)| \sin(\omega_1 t + \phi_1 + \angle H(\omega_1)) + a_2 |H(\omega_2)| \sin(\omega_2 t + \phi_2 + \angle H(\omega_2))$$

Now, before proceeding I have Two Questions

1) WHY the priciple of superposition holds for linear systems ?
2) WHY I can write $y(t)$ as in the equation above ?

If you can answer these Two important questions it means you understand Lectures 1 and 2. If not, Review Them !

(1) Has a simple answer: linear systems are governed by linear differential equations (RLC circuits) and the derivative of a linear combination of functions (input) equals the linear combination of the derivatives (output)

$$\frac{d}{dt}\left(a_1 x_1(t) + a_2 x_2(t)\right) = a_1 \frac{d}{dt} x_1(t) + a_2 \frac{d}{dt} x_2(t)$$

(2) The equation for $y(t)$ in the previous page has been derived in lecture notes #2. It shows by the superposition principle that a linear system transforms SINUSOIDS into SINUSOIDS without altering their frequency but only their amplitude and phases.

The main idea for the FOURIER SERIES relies in the superposition principle. We want to write the input signal as a superposition of sinusoids (or complex exponentials) and then apply the frequency response.

$$x(t) = \sum_{k=-\infty}^{+\infty} c_k e^{j\omega_k t} \longrightarrow \boxed{H(\omega)} \longrightarrow y(t) = \sum_{k=-\infty}^{+\infty} c_k H(\omega_k) e^{j\omega_k t}$$

In fact we have:

$$e^{j\omega_k t} = \cos \omega_k t + j \sin \omega_k t$$

and we know from the response to cosine and sine:

$$\sum_{k=-\infty}^{+\infty} c_k (\cos \omega_k t + j \sin \omega_k t) = x(t) \Rightarrow y(t) = \sum_{k=-\infty}^{+\infty} c_k |H(\omega_k)| \cos(\omega_k t + \angle H(\omega_k)) + c_k j |H(\omega_k)| \sin(\omega_k t + \angle H(\omega_k))$$

which we can also write as:

$$y(t) = \sum c_k |H(\omega_k)| e^{j\omega_k t + \angle H(\omega_k)}$$

$$= \sum c_k H(\omega_k) e^{j\omega_k t}$$

So this looks like a good strategy because if we can write $x(t)$ in the form

$$x(t) = \sum_{k=-\infty}^{+\infty} c_k e^{j\omega_k t}$$

then we can also write the output using the frequency response.

We can write $x(t)$ in this form when $x(t)$ is Periodic

For "almost" all periodic signals $x(t)$ of fundamental period $T_0$ and fundamental frequency $\omega_0 = \frac{2\pi}{T_0}$ we have:

$$\boxed{x(t) = \sum_{k=-\infty}^{+\infty} c_k e^{jk\omega_0 t}}$$

Notice that we write the frequencies $\omega_k$ as multiples $k\omega_0$ of the fundamental frequency $\omega_0$.

Now we show how to find the coefficients $c_k$ given a periodic signal $x(t)$

$$\boxed{c_n = \frac{1}{T}\int_T x(t)\, e^{-jn\omega_0 t}\, dt}$$

How do we obtein such a mysterious formula?
First let's assume:

$$x(t) = \sum_{-\infty}^{+\infty} c_k\, e^{jk\omega_0 t}$$

multiply both sides by $e^{-jn\omega_0 t}$

$$x(t)\, e^{-nj\omega_0 t} = \sum_{k=-\infty}^{+\infty} c_k\, e^{jk\omega_0 t}\, e^{-jn\omega_0 t}$$

Integrate over one period $T$

$$\int_0^T x(t)\, e^{-nj\omega_0 t}\, dt = \int_0^T \sum_{k=-\infty}^{+\infty} c_k\, e^{j(k-n)\omega_0 t}\, dt$$

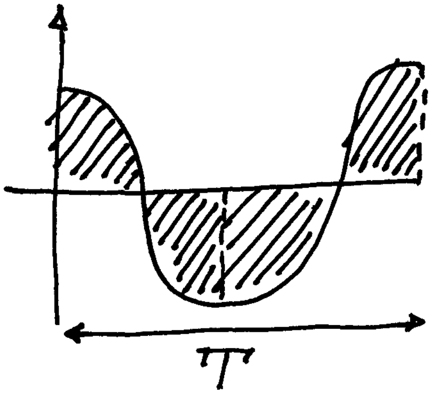
$$\underset{\text{CAREFUL WITH THIS!}}{=} \sum_{k=-\infty}^{+\infty} c_k \int_0^T e^{j(k-n)\omega_0 t}\, dt = T c_n \qquad (*)$$

The last equality follows from the following:

$$\int_0^T e^{j(k-n)\omega_0 t}\, dt = \int_0^T \cos[(k-n)\omega_0 t]\, dt + j\int_0^T \sin[(k-n)\omega_0 t]\, dt$$

$$= \begin{cases} 0 & k \neq n \\ T & k = n \end{cases}$$

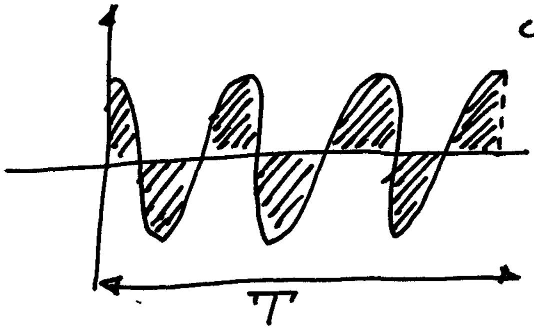
because for $k \neq n$ we integrate cosine and sin over an integer number of periods, which gives 0. For $k=0$ $\cos 0 = 1$, $\sin 0 = 0$




$\cos \omega_0 t$

The integral over the period T is zero

$\boxed{k \neq n}$




$\cos[(k-n)\omega_0 t]$

The integral is still zero as I am considering the same signal at frequency an integer multiple of $\omega_0$

$\boxed{k = n}$

For $k=n$ we have

$$\int_0^T \cos(0)\,dt + j\int_0^T \sin(0)\,dt = \int_0^T 1\,dt + j\cancel{\int_0^T 0\,dt}^{\,0} = T$$

So, putting thing together, from (*) we obtain

$$\boxed{\frac{1}{T}\int_0^T x(t)\,e^{-nj\omega_0 t}\,dt = c_n}$$

Note that it is the same if we integrate over any interval of length T and that

$$a_0 = \frac{1}{T}\int_T x(t)\,dt$$

which is called the DC - or constant component of the signal. It is the average value of the signal over one period.

We begin in the next section with a brief historical perspective in order to provide some insight into the concepts and issues that we develop in more detail in the sections and chapters that follow.

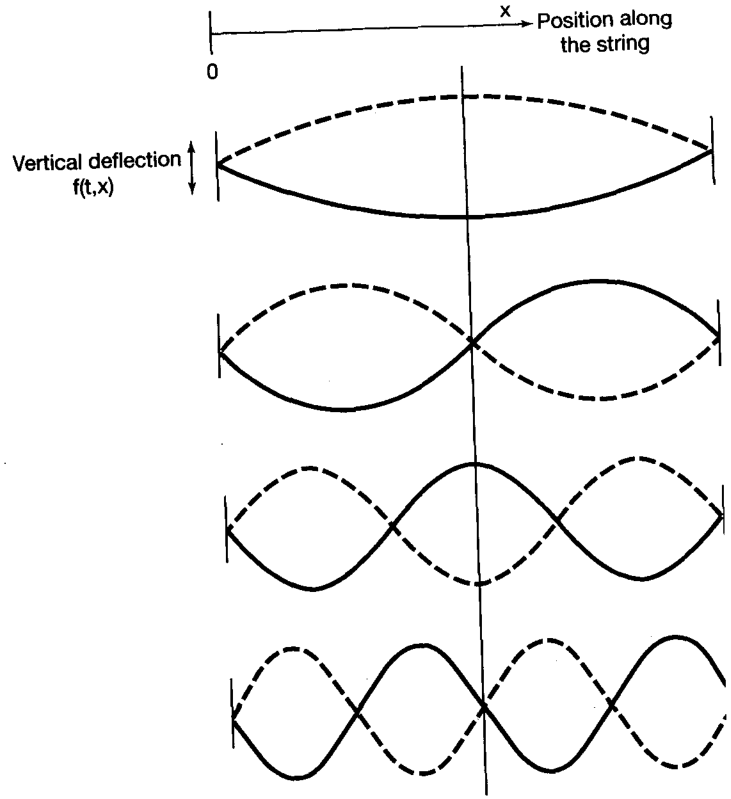
## 3.1 A HISTORICAL PERSPECTIVE

The development of Fourier analysis has a long history involving a great many individuals and the investigation of many different physical phenomena.[1] The concept of using "trigonometric sums"—that is, sums of harmonically related sines and cosines or periodic complex exponentials—to describe periodic phenomena goes back at least as far as the Babylonians, who used ideas of this type in order to predict astronomical events.[2] The modern history of the subject begins in 1748 with L. Euler, who examined the motion of a vibrating string. In Figure 3.1, we have indicated the first few of what are known as the "normal modes" of such a string. If we consider the vertical deflection $f(t, x)$ of the string at time $t$ and at a distance $x$ along the string, then for any fixed instant of time, the normal modes are harmonically related sinusoidal functions of $x$. What Euler noted was that if the configuration of a vibrating string at some point in time is a linear combination of these normal modes, so is the configuration at any subsequent time. Furthermore, Euler showed that one could calculate the coefficients for the linear combination at the later time in a very straightforward manner from the coefficients at the earlier time. In doing this, Euler performed the same type of calculation as we will in the next section in deriving one of the properties of trigonometric sums that make them so useful for the analysis of LTI systems. Specifically, we will see that if the input to an LTI system is expressed as a linear combination of periodic complex exponentials or sinusoids, the output can also be expressed in this form, with coefficients that are related in a straightforward way to those of the input.

The property described in the preceding paragraph would not be particularly useful, unless it were true that a large class of interesting functions could be represented by linear combinations of complex exponentials. In the middle of the 18th century, this point was the subject of heated debate. In 1753, D. Bernoulli argued on physical grounds that all physical motions of a string could be represented by linear combinations of normal modes, but he did not pursue this mathematically, and his ideas were not widely accepted. In fact, Euler himself discarded trigonometric series, and in 1759 J. L. Lagrange strongly criticized the use of trigonometric series in the examination of vibrating strings. His criticism was based on his own belief that it was impossible to represent signals with corners (i.e., with discontinuous slopes) using trigonometric series. Since such a configuration arises from

[1] The historical material in this chapter was taken from the following references: I. Grattan-Guiness, *Joseph Fourier, 1768–1830* (Cambridge, MA: The MIT Press, 1972); G. F. Simmons, *Differential Equations: With Applications and Historical Notes* (New York: McGraw-Hill Book Company, 1972); C. Lanczos, *Discourse on Fourier Series* (London: Oliver and Boyd, 1966); R. E. Edwards, *Fourier Series: A Modern Introduction* (New York: Springer-Verlag, 2nd ed., 1970); and A. D. Aleksandrov, A. N. Kolmogorov, and M. A. Lavrent'ev, *Mathematics: Its Content, Methods, and Meaning,* trans. S. H. Gould, Vol. II; trans. K. Hirsch, Vol. III (Cambridge, MA: The MIT Press, 1969). Of these, Grattan-Guiness' work offers the most complete account of Fourier's life and contributions. Other references are cited in several places in the chapter.

[2] H. Dym and H. P. McKean, *Fourier Series and Integrals* (New York: Academic Press, 1972). This text and the book of Simmons cited in footnote 1 also contain discussions of the vibrating-string problem and its role in the development of Fourier analysis.

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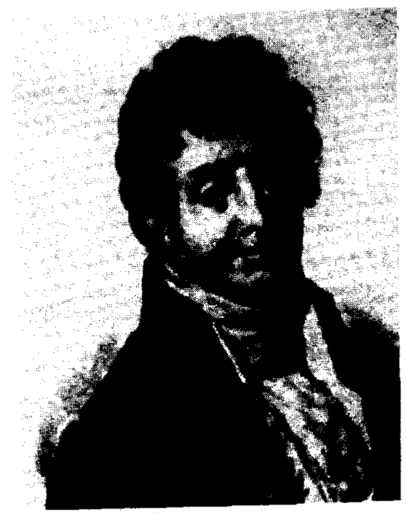
Fourier (Figure 3.2) presented his ideas half



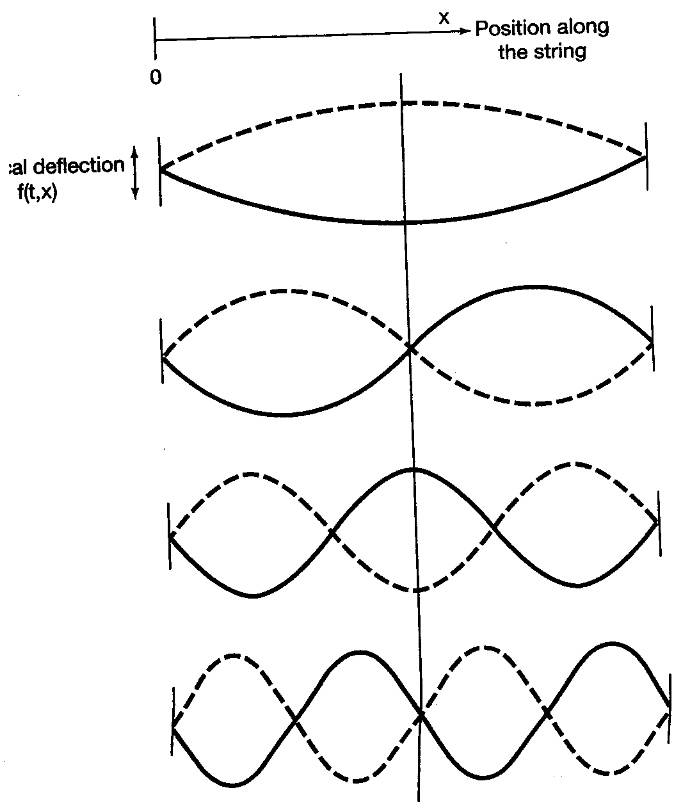



**Figure 3.1** Normal modes of a vibrating string. (Solid lines indicate the configuration of each of these modes at some fixed instant of time, $t$.)

the plucking of a string (i.e., pulling it taut and then releasing it), Lagrange argued that trigonometric series were of very limited use.

*It was in this somewhat hostile and skeptical environment that Jean Baptiste Joseph Fourier (Figure 3.2) presented his ideas half a century later. Fourier was born on March*

**Figure 3.2** Jean Baptiste Joseph

21, 1768, in Auxerre, France, and by the time of his entrance into the controversy concerning trigonometric series, he had already had a lifetime of experiences. His many contributions—in particular, those concerned with the series and transform that carry his name—are made even more impressive by the circumstances under which he worked. His revolutionary discoveries, although not completely appreciated during his own lifetime, have had a major impact on the development of mathematics and have been and still are of great importance in an extremely wide range of scientific and engineering disciplines.

In addition to his studies in mathematics, Fourier led an active political life. In fact, during the years that followed the French Revolution, his activities almost led to his downfall, as he narrowly avoided the guillotine on two separate occasions. Subsequently, Fourier became an associate of Napoleon Bonaparte, accompanied him on his expeditions to Egypt (during which time Fourier collected the information he would use later as the basis for his treatises on Egyptology), and in 1802 was appointed by Bonaparte to the position of prefect of a region of France centered in Grenoble. It was there, while serving as prefect, that Fourier developed his ideas on trigonometric series.

The physical motivation for Fourier's work was the phenomenon of heat propagation and diffusion. This in itself was a significant step in that most previous research in mathematical physics had dealt with rational and celestial mechanics. By 1807, Fourier had completed a work, Fourier had found series of harmonically related sinusoids to be useful in representing the temperature distribution through a body. In addition, he claimed that "any" periodic signal could be represented by such a series. While his treatment of this topic was significant, many of the basic ideas behind it had been discovered by others. Also, Fourier's mathematical arguments were still imprecise, and it remained for P. L. Dirichlet in 1829 to provide precise conditions under which a periodic signal could be represented by a Fourier series.[3] Thus, Fourier did not actually contribute to the mathematical theory of Fourier series. However, he did have the clear insight to see the potential for this series representation, and it was to a great extent his work and his claims that spurred much of the subsequent work on Fourier series. In addition, Fourier took this type of representation one very large step farther than any of his predecessors: He obtained a representation for *aperiodic* signals—not as weighted *sums* of harmonically related sinusoids—but as weighted *integrals* of sinusoids that are *not* all harmonically related. It is this extension from Fourier series to the Fourier integral or transform that is the focus of Chapters 4 and 5. Like the Fourier series, the Fourier transform remains one of the most powerful tools for the analysis of LTI systems.

Four distinguished mathematicians and scientists were appointed to examine the 1807 paper of Fourier. Three of the four—S. F. Lacroix, G. Monge, and P. S. de Laplace—were in favor of publication of the paper, but the fourth, J. L. Lagrange, remained adamant in rejecting trigonometric series, as he had done 50 years earlier. Because of Lagrange's vehement objections, Fourier's paper never appeared. After several other attempts to have his work accepted and published by the Institut de France, Fourier undertook the writing of another version of his work, which appeared as the text *Théorie analytique de la chaleur*.[4]

[3]Both S. D. Poisson and A. L. Cauchy had obtained results about the convergence of Fourier series before 1829, but Dirichlet's work represented such a significant extension of their results that he is usually credited with being the first to consider Fourier series convergence in a rigorous fashion.

[4]See J. B. J. Fourier, *The Analytical Theory of Heat*, trans. A. Freeman (New York: Dover, 1955).

This book was published in 1822, 15 years after Fourier had first presented his results to the Institut.

Toward the end of his life Fourier received some of the recognition he deserved, but the most significant tribute to him has been the enormous impact of his work on so many disciplines within the fields of mathematics, science, and engineering. The theory of integration, point-set topology, and eigenfunction expansions are just a few examples of topics in mathematics that have their roots in the analysis of Fourier series and integrals.[5] Furthermore, in addition to the original studies of vibration and heat diffusion, there are numerous other problems in science and engineering in which sinusoidal signals, and therefore Fourier series and transforms, play an important role. For example, sinusoidal signals arise naturally in describing the motion of the planets and the periodic behavior of the earth's climate. Alternating-current sources generate sinusoidal voltages and currents, and, as we will see, the tools of Fourier analysis enable us to analyze the response of an LTI system, such as a circuit, to such sinusoidal inputs. Also, as illustrated in Figure 3.3, waves in the ocean consist of the linear combination of sinusoidal waves with different spatial periods or *wavelengths*. Signals transmitted by radio and television stations are sinusoidal in nature as well, and as a quick perusal of any text on Fourier analysis will show, the range of applications in which sinusoidal signals arise and in which the tools of Fourier analysis are useful extends far beyond these few examples.

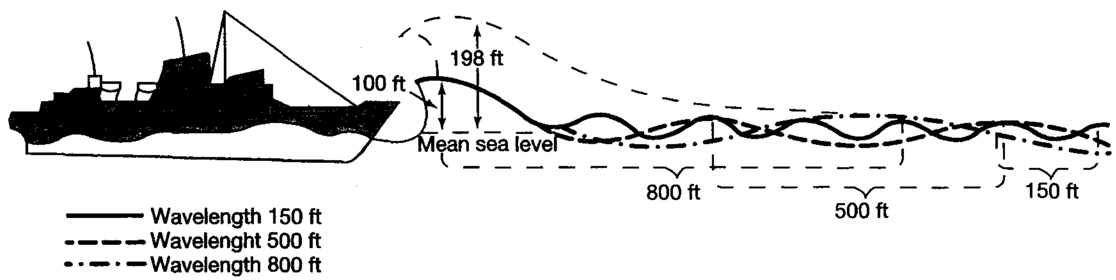



**Figure 3.3** Ship encountering the superposition of three wave trains, each with a different spatial period. When these waves reinforce one another, a very large wave can result. In more severe seas, a giant wave indicated by the dotted line could result. Whether such a reinforcement occurs at any location depends upon the relative phases of the components that are superposed. [Adapted from an illustration by P. Mion in "Nightmare Waves Are All Too Real to Deepwater Sailors," by P. Britton, *Smithsonian* 8 (February 1978), pp. 64–65].

While many of the applications in the preceding paragraph, as well as the original work of Fourier and his contemporaries on problems of mathematical physics, focus on phenomena in continuous time, the tools of Fourier analysis for discrete-time signals and systems have their own distinct historical roots and equally rich set of applications. In particular, discrete-time concepts and methods are fundamental to the discipline of numerical analysis. Formulas for the processing of discrete sets of data points to produce numerical approximations for interpolation, integration, and differentiation were being investigated as early as the time of Newton in the 1600s. In addition, the problem of predicting the motion of a heavenly body, given a sequence of observations of the body, spurred the investigation of harmonic time series in the 18th and 19th centuries by eminent scientists and mathematicians, including Gauss, and thus provided a second setting in which much of the initial work was done on discrete-time signals and systems.

In the mid-1960s an algorithm, now known as the fast Fourier transform, or FFT, was introduced. This algorithm, which was independently discovered by Cooley and Tukey in 1965, also has a considerable history and can, in fact, be found in Gauss' notebooks.[6] What made its modern discovery so important was the fact that the FFT proved to be perfectly suited for efficient digital implementation, and it reduced the time required to compute transforms by orders of magnitude. With this tool, many interesting but previously impractical ideas utilizing the discrete-time Fourier series and transform suddenly became practical, and the development of discrete-time signal and system analysis techniques moved forward at an accelerated pace.

What has emerged out of this long history is a powerful and cohesive framework for the analysis of continuous-time and discrete-time signals and systems and an extraordinarily broad array of existing and potential applications. In this and the following chapters, we will develop the basic tools of that framework and examine some of its important implications.

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