ECE 45

Prof. Franceschetti

LECTURE NOTES #5

FOURIER SERIES

In last lecture we have seen a way to represent periodic signals in terms of Fourier Series, which is a convenient representation to find the response of the system using the principle of superposition and the frequency response.

What we have done:

(1) Assume we can write $\boxed{x(t) = \sum_{k=-\infty}^{+\infty} c_k e^{jk\omega_0 t}}$

$\omega_0 = \frac{2\pi}{T}$ is the fundamental frequency of the periodic signal $x(t)$

$e^{jk\omega_0 t}$ are the harmonic components at frequency $k\omega_0$ of the signal $x(t)$

c_k are the coefficients of the series expansion.

(2) We have found a formula to compute the coefficient c_n $\boxed{c_n = \frac{1}{T}\int_0^T x(t)\, e^{-nj\omega_0 t}\, dt}$

The Avg power of the signal is conserved in its harmonic components

PARSEVAL THEOREM

$$\frac{1}{T}\int_T |x(t)|^2 dt = \sum_{k=-\infty}^{+\infty} |c_k|^2$$

Proof

$$|x(t)|^2 = x(t)\, x^*(t)$$

$$x(t) = \sum_{k=-\infty}^{+\infty} c_k\, e^{jk\omega_0 t}$$

$$x^*(t) = \sum_{k=-\infty}^{+\infty} c_k^*\, e^{-jk\omega_0 t}$$

$$\frac{1}{T}\int |x(t)|^2 dt = \frac{1}{T}\int_T \sum_{k=-\infty}^{+\infty} c_k e^{jk\omega_0 t} \cdot \sum_{m=-\infty}^{+\infty} c_m^* e^{-jm\omega_0 t}\, dt$$

$$= \frac{1}{T}\int_T \sum_k \sum_m c_k c_m^* e^{j(k-m)\omega_0 t}\, dt$$

$$= \frac{1}{T}\int_T \sum_k c_k c_k^* + \sum_k \sum_{m\neq k} c_k c_m^* e^{j(k-m)\omega_0 t}\, dt$$

$$= \sum_{k=-\infty}^{+\infty} |c_k|^2 + \sum_k \sum_{m\neq k} \frac{c_k c_m^*}{T} \cancel{\int_T e^{j(k-m)\omega_0 t}\, dt} \to 0$$

Notice that in the application of the proof of Parseval Theorem we have discovered an important property of the Fourier coefficients. That is:

If a signal $x(t)$ has Fourier coeff. c_k
then $x(t)^*$ has Fourier coeff. c_{-k}^*

From which we also have
if $x(t)$ is real then $x(t) = x^*(t)$
so that $\boxed{c_k = c_{-k}^*}$

A similar property is Time reversal:
If a signal $x(t)$ has Fourier coeff. c_k
then $x(-t)$ has Fourier coeff. c_{-k}

From which we also have:

if $x(t)$ is even $x(t) = x(-t)$
then $\boxed{c_k = c_{-k}}$

if $x(t)$ is odd $x(t) = -x(-t)$
then $\boxed{c_k = -c_{-k}}$

Another property that we can see from the proof of Parseval Theorem is the <u>multiplication property</u>

$$x(t) = \sum_k c_k e^{jk\omega_0 t}$$

$$y(t) = \sum_m c'_m e^{jm\omega_0 t}$$

$$x(t)\,y(t) = \sum_k \sum_m c_k c'_m e^{j(k+m)\omega_0 t}$$

let $k+m = l$
$m = l - k$

$$x(t)\,y(t) = \sum_{k=-\infty}^{+\infty} \sum_{l=-\infty}^{+\infty} c_k c'_{l-k}\, e^{jl\omega_0 t}$$

$$= \sum_{l=-\infty}^{+\infty} \underbrace{\sum_k c_k c'_{l-k}}\; e^{jl\omega_0 t}$$

$$\boxed{\bar{c}_l = \sum_{k=-\infty}^{+\infty} c_k c'_{l-k}}$$

The Fourier coeff. of the product of two Fourier Series are given by the above expression which is called the discrete convolution of the coefficients

Finally, we have the time-shift property

$$x(t) = \sum_{k=-\infty}^{+\infty} c_k \, e^{jk\omega_0 t}$$

$$y(t) = x(t-t_0) = \sum_{k=-\infty}^{+\infty} c_k \, e^{jk\omega_0 (t-t_0)}$$

$$= \sum_{k=-\infty}^{+\infty} c_k \, e^{jk\omega_0 t} \, e^{-jk\omega_0 t_0}$$

$$=$$

if we shift the signal in Time
The coeff. c_k has a phase shift

And the time-scaling

$$x(t) = \sum_{k=-\infty}^{+\infty} c_k \, e^{jk\omega_0 t}$$

$$y(t) = x(at) = \sum c_k \, e^{jk \underbrace{\omega_0 a}_{} t}$$

freq. is scaled by the same factor

You might have noticed (and I told you to be careful!) that in the computation of the coefficients of the Fourier series and in some of the properties we are freely exchanging the order of the series and integral or the order of two series.

This can be done only if we assume a certain kind of convergence of the Fourier series to the periodic function $x(t)$ and has been source of debate of mathematician for many years.
Let's see what we are doing.

- We are assuming we can write a periodic signal $x(t) = \sum_{k=-\infty}^{+\infty} c_k e^{jk\omega_0 t}$ which means that the series converges to $x(t)$ in some well defined sense.

- Let $c_k e^{jk\omega_0 t} = f(k,t)$, in the computations we are often doing the following:

$$\int_T \sum_k f(k,t) = \sum_k \int_T f(k,t)\,dt$$

This is allowed only if the series converges in some well defined sense.
More precisely, a Theorem by Fubini states that this inversion operation is allowed if:

$$\frac{1}{T}\int_T \sum_{k=-\infty}^{+\infty} |c_k e^{jk\omega_0 t}|\,dt < \infty$$

$$= \frac{1}{T}\int_T \sum |c_k|\,dt = \frac{\cancel{T}}{\cancel{T}} \sum_{k=-\infty}^{+\infty} |c_k| < \infty$$

which means that:

$$\frac{1}{T}\int_T |x(t)|\,dt = \frac{1}{T}\int_T \left| \sum c_k e^{jk\omega_0 t} \right| dt \le \frac{1}{T}\int_T \sum |c_k e^{jk\omega_0 t}|\,dt = \sum_{k=-\infty}^{+\infty} |c_k| < \infty$$

Notice that we have said that in order to invert the integral and the series it is sufficient (by Fubini)

$$\sum |c_k| < \infty$$

which implies

$$\boxed{\frac{1}{T}\int_T |x(t)|\,dt < \infty}$$ absolute integrability over a period.

is this latter condition enough to guarantee the exchange and that the series converges? In general <u>not</u>.
We have to add two more conditions:

DIRICHLET CONDITIONS

Any periodic signal $x(t)$ has a Fourier series which converges to $x(t)$ for all t except where $x(t)$ is discontinuous where it converges to the average of the two values at either side of the discontinuity if:

1) $x(t)$ is absolutely integrable over a period
2) $x(t)$ has a finite number of discontinuities over any finite interval
3) $x(t)$ has a finite number of max and min over any finite interval.

However, we have another condition which implies Dirichlet condition 1) and is a finite energy condition which guarantees "weak convergence" of the series.

Let's look again at condition 1 which is the most important.
We have:

$$\frac{1}{T}\left|\int_T |x(t)|\,dt\right|^2 \leq \frac{1}{T}\int_T |x(t)|^2\,dt$$

So if

$$\frac{1}{T}\int_T |x(t)|^2\,dt < \infty$$

$\boxed{\text{Finite Average power over a period}}$

$\Downarrow$

this implies

$$\frac{1}{T}\left|\int_T |x(t)|\,dt\right| < \infty$$

$\boxed{\text{Absolute integrability over a period}}$ Dirichlet condition (1)

If only the finite power (energy) conditions is satisfied, we have that the Fourier series converges in Energy to x(t) although it might not converge in every point:

$$\left| x(t) - \sum_{k=-\infty}^{+\infty} c_k e^{jk\omega_0 t}\right|$$ might not be 0 for all t but:

$$\lim_{M\to\infty} \int_T \left| x(t) - \sum_{k=-M}^{M} c_k e^{jk\omega_0 t}\right|^2 dt = 0$$

$\boxed{\text{The energies of the two signals are the same at every point.}}$

ECE 45 ADDENDUM for LECTURE 5

Let any signal $x(t)$

Define: $|x^*(t)|^2$ The instantaneous power at time t

$$\int_T |x^*(t)|^2 \, dt$$ the energy of the signal over interval T

$$\frac{1}{T}\int_T |x(t)|^2 \, dt$$ the average power over the interval T