ECE 45

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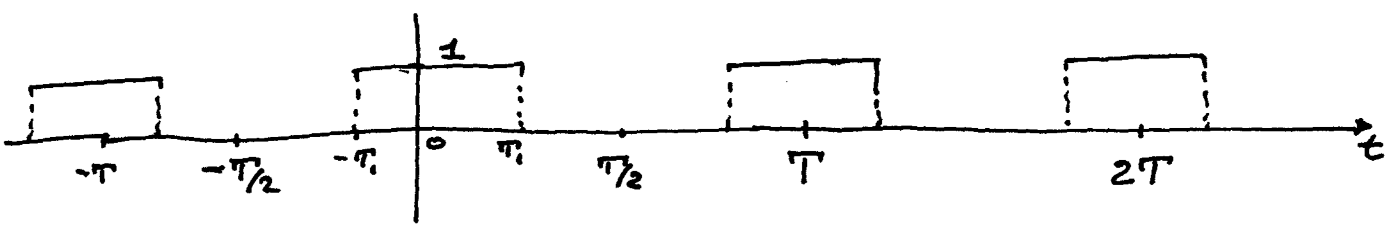
Lecture notes #6

FOURIER TRANSFORM

The Fourier Transform is the natural extension of the Fourier Series representation to signals that are non-periodic and of finite energy.

To introduce the Fourier Transform let us start with an example and consider the Fourier Series of the following periodic signal:



$$c_0 = \frac{1}{T}\int_{-T/2}^{T/2} 1\, dt = \frac{2T_1}{T}$$

$$\underset{k\neq 0}{c_k} = \frac{1}{T}\int_{-T_1}^{T_1} e^{-jk\omega_0 t}\, dt = -\frac{1}{jk\omega_0 T}\left[e^{-jk\omega_0 t}\right]_{-T_1}^{T_1}$$

$$= \frac{1}{k\omega_0 T}\;\frac{e^{jk\omega_0 T_1} - e^{-jk\omega_0 T_1}}{j} =$$

$$= \frac{2}{k\omega_0 T}\;\sin(k\omega_0 T_1) =$$

$$= \boxed{\frac{\sin k\omega_0 T_1}{k\pi}}$$

So we have:

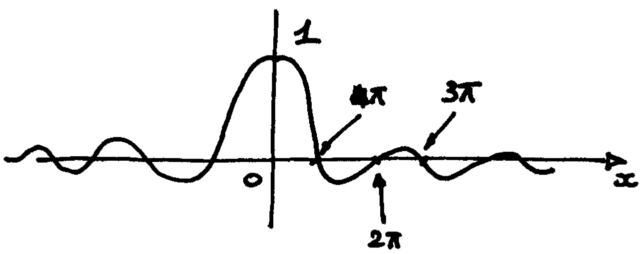
$$c_0 T = 2T_1$$

$$c_k T = 2\frac{\sin(k\omega_0 T_1)}{k\omega_0} = 2\left.\frac{\sin(\omega T_1)}{\omega}\right|_{\omega=k\omega_0} = 2T_1 \left.\frac{\sin\omega T_1}{\omega T_1}\right|_{\omega=k\omega_0}$$

Let's define the sinc function as:

$\mathrm{sinc}(x) = \dfrac{\sin x}{x}$ This function

has the following plot:

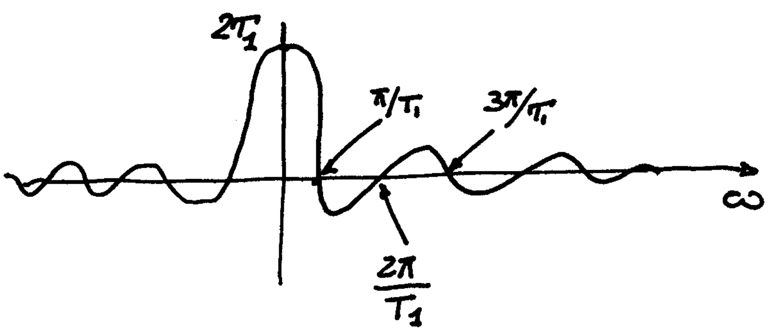


The zeros of this function are at integer multiples of π and the value in $x=0$ is 1

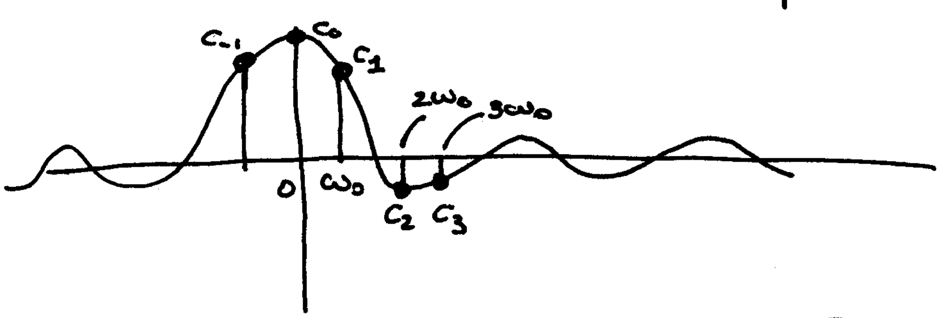
Now, the function we obtained for the Fourier coefficients looks very much like this function:

$$\boxed{2T_1 \frac{\sin\omega T_1}{\omega T_1}}$$

just substitute x with ωT_1



We can see the Fourier coefficients c_k as "sampled points" of the above function at $\omega = k\omega_0 = k\frac{2\pi}{T}$



Notice something important:

1) The zeros of the function depend on T_1, the width of the rectangle. The wider the rectangles in square wave are the thinner the sinc pulse appears

2) The maximum of the function is proportional to T_1, the wider the rectangles in the square wave are, the higher the sinc pulse appears.

3) The sampled points of the function, which represent the Fourier coefficients depend on the period of the rectangular square wave. The larger the period T is the closer the samples are.

4) As $\omega_0 \rightarrow 0$ or equivalently $T \rightarrow \infty$, the samples representing the coefficients c_k become very close to each other and their discrete sequence resembles more and more the continuous function.

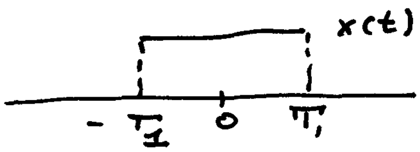
QUESTION: What does it mean $\omega_0 \rightarrow 0$ for the original signal in time domain?
The signal tends to "loose its periodicity" as the periodic cycle T becomes larger and larger

So we have two effects:

1. the coeff $c_k = 2T_1 \dfrac{\sin k\omega_0 T_1}{k\omega_0 T_1}$ become to resemble the continuous function

$$\boxed{2T_1 \frac{\sin \omega T_1}{\omega T_1}}$$

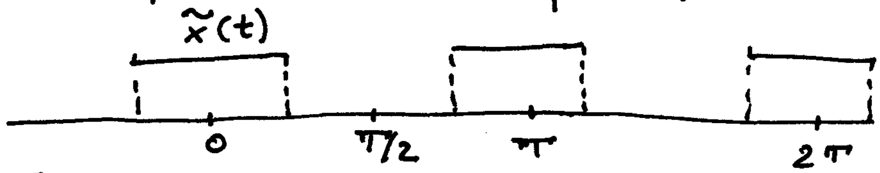
2. The signal tends to become a non-periodic rectangular function



We <u>define</u> the Fourier Transform of $x(t)$ as:

$$FT(x(t)) = X(j\omega) = \int_{-\infty}^{+\infty} x(t)\, e^{-j\omega t}\, dt$$

Now, consider the periodic extension of $x(t)$



What is the Fourier Series representation of $\tilde{x}(t)$?

$$c_k = \frac{1}{T}\int_{-T/2}^{T/2} \tilde{x}(t)\, e^{-jk\omega_0 t}\, dt = \frac{1}{T}\int_{-\infty}^{+\infty} x(t)\, e^{-jk\omega_0 t}\, dt$$

$$= \boxed{\frac{1}{T} X(jk\omega_0)}$$

$$\tilde{x}(t) = \sum_{k=-\infty}^{+\infty} c_k\, e^{jk\omega_0 t} = \sum_{k=-\infty}^{+\infty} \frac{1}{T} X(jk\omega_0)\, e^{jk\omega_0 t} = \frac{1}{2\pi} \sum_{k=-\infty}^{+\infty} X(jk\omega_0)\, e^{jk\omega_0 t}\, \omega_0$$

$\frac{2\pi}{T} = \omega_0$

So putting things together, we have

$$\boxed{\tilde{x}(t) = \frac{1}{2\pi} \sum_{k=-\infty}^{+\infty} X(jk\omega_0)\, e^{jk\omega_0 t}\, \omega_0} \qquad (1)$$

This is the F.S. representation of the periodic signal $\tilde{x}(t)$
The coefficients c_k are related to the F.T. of its periodic extension :
$$\boxed{c_k = \frac{1}{T} X(jk\omega_0)}$$

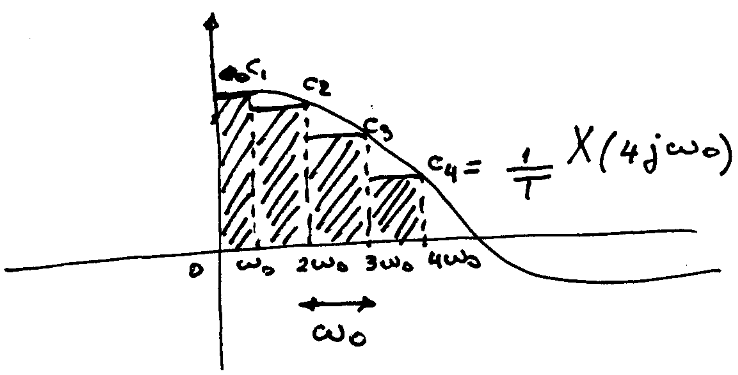
As $T \to \infty$, $\omega_0 \to 0$
we have :

$$\tilde{x}(t) \longrightarrow x(t)$$

and from (1)

$$\boxed{x(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} X(j\omega)\, e^{j\omega t}\, d\omega}$$

Question : why did we substitute the sum in (1) with an integral as $\omega_0 \to 0$? That sum is called a Riemann Sum and converges to an integral in the limit $\omega_0 \to 0$



The area of each little rectangle is
$\omega_0 \cdot X(jk\omega_0)$

The sum of the areas

$$\sum_k \omega_0 X(jk\omega_0) e^{jk\omega_0 t} \longrightarrow \int X(j\omega) e^{j\omega t}\, dt$$

In summary,

we have two important formulas for non-periodic signals:

$$\boxed{x(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} X(j\omega)\, e^{j\omega t}\, d\omega}$$ inverse Fourier Transform

where

$$\boxed{X(j\omega) = \int_{-\infty}^{+\infty} x(t)\, e^{-j\omega t}\, dt}$$ Fourier Transform

FINALLY : why is this useful?

Remember the frequency response $H(\omega)$. This represented the response of the system at a given frequency ω. By the superposition principle we computed the response to a periodic signal $f(t)$ as:

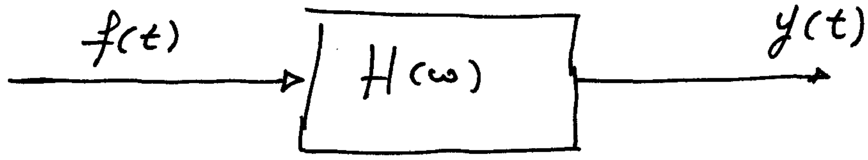
$$y(t) = \sum_{k=-\infty}^{+\infty} c_k\, H(k\omega_0)\, e^{jk\omega_0 t}$$

(superposition of the responses to the individual frequencies $k\omega_0$)

Now we have a similar situation, only we have a SPECTRUM of frequencies ω rather than individual frequencies $k\omega_0$

$$y(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} X(j\omega)\, H(\omega)\, e^{j\omega t}\, d\omega$$

PERIODIC CASE



$$f(t) = \sum_{k=-\infty}^{+\infty} c_k \, e^{jk\omega_0 t}$$

$$c_k = \frac{1}{T}\int_T f(t)\, e^{-jk\omega_0 t}\, dt$$

$$y(t) = \sum_{k=-\infty}^{+\infty} c_k \, H(\omega)\, e^{jk\omega_0 t}$$

notice that the coeff c_k plays the role of the FT $X(j\omega)$

NON - PERIODIC CASE

$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} X(j\omega)\, e^{j\omega t}\, d\omega$$

$$y(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} H(\omega)\, X(j\omega)\, e^{j\omega t}\, d\omega$$

$$X(j\omega) = \int_{-\infty}^{+\infty} f(t)\, e^{-j\omega t}\, dt$$