FOURIER TRANSFORM
(continued)

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} X(j\omega)\, e^{j\omega t}\, d\omega$$

$$X(j\omega) = \int_{-\infty}^{+\infty} x(t)\, e^{-j\omega t}\, dt$$

We have obtained this representation as the generalization of the Fourier Series for non-periodic signals.
As with the Fourier Series there is an issue of <u>convergence</u> for when can we use the above representation.
<u>If we are satisfied</u> by having no energy in the error between the Fourier representation and the original signal, then we need:

$$\boxed{\int_{-\infty}^{+\infty} |x(t)|^2\, dt < \infty}$$

and in this case we have that:

① $X(j\omega) = \int_{-\infty}^{+\infty} x(t)\, e^{-j\omega t}\, dt < \infty$

② $\int_{-\infty}^{+\infty} |err(t)|^2\, dt = 0$

where: $err(t) = x(t) - \hat{x}(t)$

$$\hat{x}(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} X(j\omega)\, e^{j\omega t}\, d\omega .$$

If instead we want point-wise convergence between the Fourier representation and the original signal, then we need:

DIRICHLET CONDITIONS

(1) $x(t)$ is absolutely integrable: $\int_{-\infty}^{+\infty} |x(t)|\,dt < \infty$

(2) $x(t)$ has a finite number of max and minima in any finite interval

(3) $x(t)$ has a finite number of discontinuities that are finite

Given the three conditions above, $x(t)$ and $\hat{x}(t)$ are the same at every point except at discontinuities where $\hat{x}(t)$ is the average of left and right limit points of $x(t)$

Notice that the discussion of convergence above is the same of the one we already had for the Fourier Series. So, it is a good time now to compare these notes with the ones of the Fourier Series.

Next, we look at some useful properties of the Fourier Transform.

PARSEVAL THEOREM

" The power is conserved in the spectrum"

$$\int_{-\infty}^{+\infty} |x(t)|^2 dt = \int_{-\infty}^{+\infty} x(t)\, x^*(t)\, dt$$

$$= \int_{-\infty}^{+\infty} x(t) \frac{1}{2\pi} \int_{-\infty}^{+\infty} X^*(j\omega)\, e^{-j\omega t}\, d\omega\, dt$$

$$= \frac{1}{2\pi} \int_{-\infty}^{+\infty} X^*(j\omega) \int_{-\infty}^{+\infty} x(t)\, e^{-j\omega t}\, dt\, d\omega$$

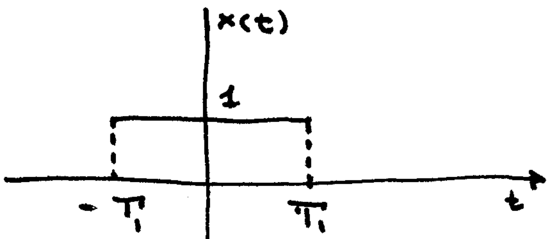
$$= \frac{1}{2\pi} \int_{-\infty}^{+\infty} X^*(j\omega)\, X(j\omega)\, d\omega$$

$$= \frac{1}{2\pi} \int_{-\infty}^{+\infty} |X(j\omega)|^2\, d\omega$$

So we can equivalently compute the power of the signal by performing integration in the time or in the frequency domain.

TIME - FREQUENCY duality

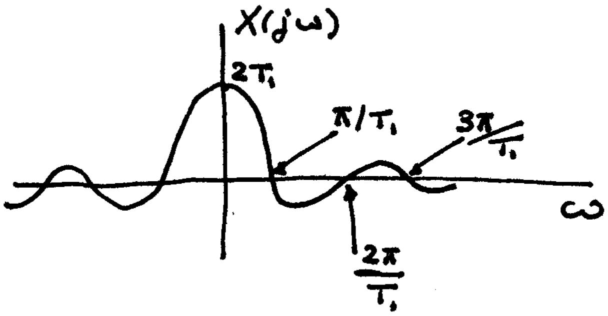
We illustrate this concept through an example.
Let's recall the FT of a rectangular signal:



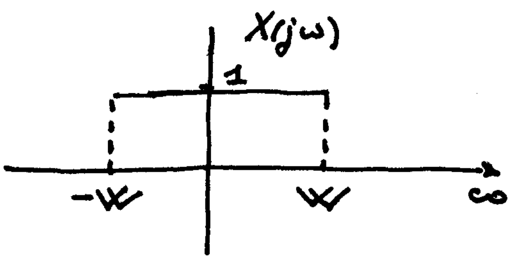
$$X(j\omega) = \int_{-T_1}^{T_1} e^{-j\omega t}\, dt = \left[\frac{e^{-j\omega t}}{-j\omega} \right]_{-T_1}^{T_1}$$

$$= \frac{e^{-j\omega T_1} - e^{j\omega T_1}}{-j\omega} = \frac{e^{j\omega T_1} - e^{-j\omega T_1}}{j\omega}$$

$$X(j\omega) = 2T_1 \frac{\sin \omega T_1}{\omega T_1}$$



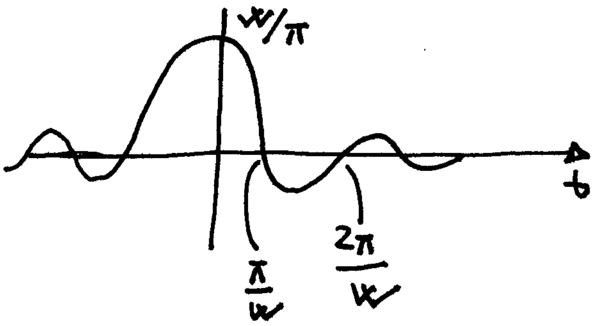
Next, let's compute the inverse transform of a rectangular signal in the frequency domain. We have:



$$x(t) = \frac{1}{2\pi} \int_{-W}^{W} e^{j\omega t}\, d\omega = \frac{1}{2\pi} \left[\frac{e^{j\omega t}}{jt} \right]_{-W}^{W}$$

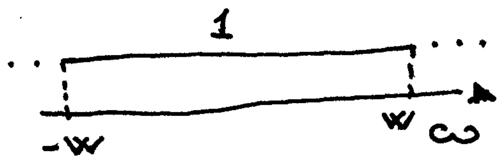
$$= \frac{1}{2\pi} \frac{e^{jWt} - e^{-jWt}}{jt} = \frac{\sin Wt}{\pi t}$$

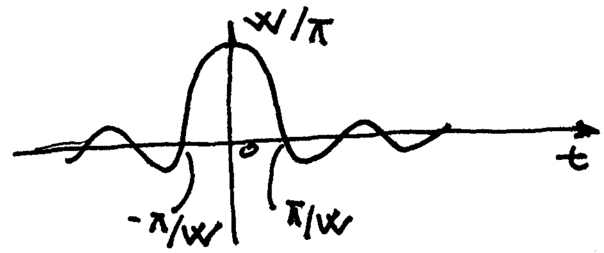
$$= \frac{W}{\pi} \sin \frac{Wt}{Wt}$$



So, we have discovered that the inverse transform of a rectangle in the frequency domain is also a sinc function

Now, if we let $W \to \infty$ we have that $X(j\omega)$ is a very long rectangle in the frequency domain (it approaches a constant)





At the same time the sinc function tends to become an "impulsive" function centered at the origin in the time domain. We call this impulsive function (which we will define more formally later) a δ-function $\delta(t)$.

So we have that in the limit:

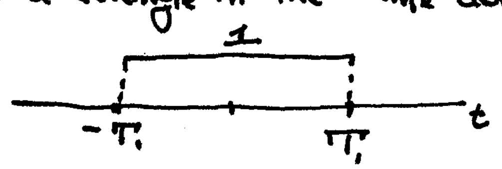
$FT^{-1}(1) = \delta(t)$

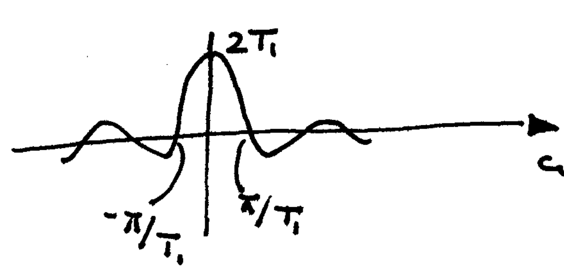
$FT(\delta(t)) = 1$

Similarly, we have starting with a rectangle in the time domain and letting $T_1 \to \infty$

$FT(1) = 2\pi\delta(\omega)$

$FT^{-1}(2\pi\delta(\omega)) = 1$





Notice that we have multiplied by 2π to normalize the sinc function properly.

Summarizing. We have defined the impulse $\delta(t)$ as

$$\boxed{\delta(t) = \lim_{W\to\infty} \frac{W}{\pi} \frac{\sin Wt}{Wt}}$$

with this definition we have that $FT(\delta(t)) = 1$

Similarly, we have

$$\boxed{2\pi\,\delta(\omega) = \lim_{T_1\to\infty} 2T_1 \frac{\sin \omega T_1}{\omega T_1}}$$

and therefore $FT(1) = 2\pi\delta(\omega)$

The Fourier Transform of a constant contains only the frequency component at the origin, as the impulse $\delta(\omega)$ is a "spike" at $\omega=0$

The Fourier Transform of an impulse $\delta(t)$ contains all frequency components as it is constant over the whole frequency spectrum.

Finally, notice

$$FT(\delta(t)) = \int_{-\infty}^{+\infty} e^{-j\omega t}\,\delta(t)\,dt = 1$$

$$FT^{-1}(2\pi\,\delta(\omega)) = \frac{1}{\cancel{2\pi}}\int_{-\infty}^{+\infty} \cancel{2\pi}\,\delta(\omega)\,e^{j\omega t}\,d\omega = 1$$

Before talking in more detail about the δ-function, let's look at some other properties of the Fourier Transform.

TIME SHIFT

$$x(t-t_0) \longleftrightarrow X(j\omega)\, e^{-j\omega t_0}$$

Proof

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} X(j\omega)\, e^{j\omega t}\, d\omega$$

$$x(t-t_0) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} X(j\omega)\, e^{j\omega(t-t_0)}\, d\omega$$

$$= \frac{1}{2\pi}\int_{-\infty}^{+\infty} X(j\omega)\, e^{-j\omega t_0}\, e^{j\omega t}\, d\omega$$

$$= FT^{-1}\left(X(j\omega)\, e^{-j\omega t_0} \right)$$

So we have that a shift in the time domain corresponds by a phase shift in the frequency domain

CONJUGATION

$$x^*(t) \longleftrightarrow X^*(-j\omega)$$

Proof

$$X^*(j\omega) = \left(\int_{-\infty}^{+\infty} x(t)\, e^{-j\omega t}\, dt \right)^* = \int_{-\infty}^{+\infty} x^*(t)\, e^{j\omega t}\, dt$$

$$X^*(-j\omega) = \int_{-\infty}^{+\infty} x^*(t)\, e^{-j\omega t}$$

DIFFERENTIATION

$$\frac{d\,x(t)}{dt} \longleftrightarrow j\omega\, X(j\omega)$$

Proof

$$\frac{d\,x(t)}{dt} = \frac{1}{2\pi}\int_{-\infty}^{+\infty} j\omega\, X(j\omega)\, e^{j\omega t}\, d\omega$$

$$= \frac{1}{2\pi}\int_{-\infty}^{+\infty} FT\left(\frac{d\,x(t)}{dt}\right) e^{j\omega t}\, d\omega$$

TIME and FREQUENCY SCALING

$$x(at) \longleftrightarrow \frac{1}{|a|}\, X\left(j\frac{\omega}{a}\right)$$

Proof

$$\int_{-\infty}^{+\infty} x(at)\, e^{-j\omega t}\, dt$$

let $\tau = at$

$d\tau = a\, dt$

$\frac{d\tau}{a} = dt$

for $(a>0)$ $\quad \frac{1}{a}\int_{-\infty}^{+\infty} x(\tau)\, e^{-j\omega\frac{\tau}{a}}\, d\tau$

for $(a<0)$ $\quad \frac{1}{a}\int_{+\infty}^{-\infty} x(\tau)\, e^{-j\omega\tau/a}\, d\tau = -\frac{1}{a}\int_{-\infty}^{+\infty} x(\tau)\, e^{-j\omega\frac{\tau}{a}}\, d\tau$