ECE 45
Prof. Franceschetti

DIRAC's δ-function

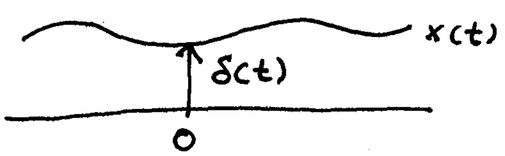
In last lecture we have introduced the δ-function as the limiting function of the sinc as this becomes an impulse centered at the origin.
In this lecture we take a closer look.

Let's first give a formal definition and then we give some intuition for it.

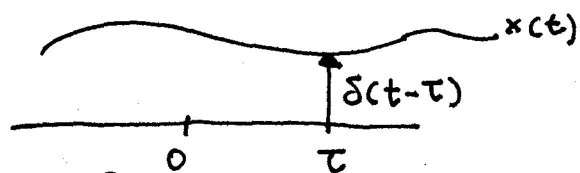
DEFINE $\delta(t)$ a function such that for any continuous signal $x(t)$, we have:

$$\int_{t_1}^{t_2} x(t)\,\delta(t)\,dt = \begin{cases} x(0) & \text{if } 0 \in [t_1, t_2] \\ 0 & \text{if } 0 \in [t_1, t_2] \end{cases}$$

and we indicate it graphically with an arrow (indicating the "spike" of the impulse.



$$\int_{-\infty}^{+\infty} x(t)\,\delta(t)\,dt = x(0)$$



$$\int_{-\infty}^{+\infty} x(t)\,\delta(t-\tau)\,dt = x(\tau)$$

We also have another important property.

CONVOLUTION PROPERTY of the δ-function

$$\boxed{\int_{-\infty}^{+\infty} x(\tau)\,\delta(t-\tau)\,d\tau = x(t)}$$

the integral $\int_{-\infty}^{+\infty} f(\tau)\,g(t-\tau)\,d\tau = f \circledast g\,(t)$

is called the convolution integral, so the property above says that convolution of the signal x withe the δ-function is the signal $x(t)$ itself. In other words, convolution with the δ-function is equivalent to letting the signal pass-through undisturbed.

Let's prove the property:

$$\int_{-\infty}^{+\infty} x(t)\,\delta(t-\tau)\,dt = x(\tau)$$

call $t \to \tau$ $\quad \tau \to t$, we have

$$\int_{-\infty}^{+\infty} x(\tau)\,\delta(\tau-t)\,d\tau = x(t)$$

But we also have that $\delta(t)$ is an even-function, so $\delta(t-\tau) = \delta(\tau-t)$ and we finally have:

$$\int_{-\infty}^{+\infty} x(\tau)\,\delta(t-\tau)\,d\tau = x(t).$$

The fact that $\delta(t)$ is even follows by the following

SCALING PROPERTY

$$\boxed{\delta(ta) = \frac{1}{|a|}\,\delta(t)}$$

which implies $\delta(t) = \delta(-t)$

Let's prove the scaling property:

$$\int_{-\infty}^{+\infty} x(t)\,\delta(at)\,dt \underset{\substack{\text{call } t' = at \\ dt' = a\,dt}}{=} \int x\left(\frac{t'}{a}\right)\delta(t')\,\frac{dt'}{a} = \begin{cases} \boxed{a>0} \displaystyle\int_{-\infty}^{+\infty} \frac{1}{a}\,x\left(\frac{t'}{a}\right)\delta(t')\,dt' \\ \boxed{a<0} \displaystyle\int_{+\infty}^{-\infty} \frac{1}{a}\,x\left(\frac{t'}{a}\right)\delta(t')\,dt' \end{cases}$$

$$= \frac{1}{|a|}\int_{-\infty}^{+\infty} x\left(\frac{t'}{a}\right)\delta(t')\,dt' = \frac{1}{|a|}\,x(0)$$

so now the property follows by the definition of δ-function.

We have proven the convolution property of the δ-function: convolution of a signal by $\delta(t)$ is the signal itself. Let's see what does this mean in the frequency domain:

$$\boxed{x \circledast h(t) \longleftrightarrow X(j\omega)\cdot H(j\omega)}$$

The convolution integral in the time-domain corresponds to a simple multiplication in the frequency domain. So, by this property, we have:

$$FT\left(x \circledast \delta(t)\right) = X(j\omega)\cdot FT(\delta(t)) = X(j\omega)\cdot 1$$

since: $FT(\delta(t)) = \displaystyle\int_{-\infty}^{+\infty} \delta(t)\,e^{-j\omega t}\,dt = e^{0} = 1$

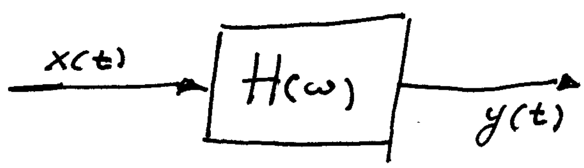
Summarizing, we have:

convolution in time domain corresponds to multiplication in freq. domain

The FT of $\delta(t)$ is a constant

$\Rightarrow$ convolution in time-domain with $\delta(t)$ corresponds to multiplication by a constant 1 in the frequency domain.

Now, a <u>simple exercise</u>



Let $x(t) = \delta(t)$ and let's solve this system

$FT(x(t)) = 1$

$Y(j\omega) = 1 \cdot H(\omega)$

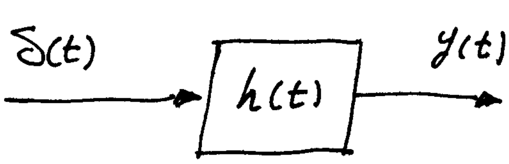
$$\boxed{y(t) = FT^{-1}[H(\omega)]}$$

so the inverse FT of the function $H(\omega)$ is the <u>response of the system</u> to the input $\delta(t)$

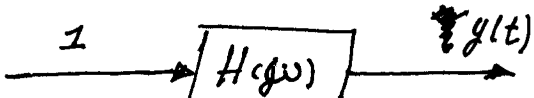
But by the convolution property, we also have:

$$FT^{-1}[1 \cdot H(\omega)] = \boxed{\delta \circledast h(t) = y(t)}$$

so the <u>response of the system to the $\delta(t)$</u> is given by the convolution of $\delta(t)$ with $h(t)$, the inverse FT of $H(\omega)$.



$$y(t) = \int_{-\infty}^{+\infty} h(t)\,\delta(t-\tau)\,d\tau = h(t)$$



$$y(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} H(\omega)e^{j\omega t}\,d\omega = h(t)$$

So we have found that our "favorite function" $H(\omega)$ that we have used many times to study linear systems and to solve problem sets and which was introduced as

$H(\omega)$

"The response of the system to ~~the~~ an input at a single frequency ω" (1)

is also the Fourier Transform of

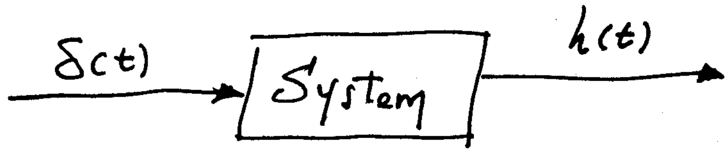
$h(t)$

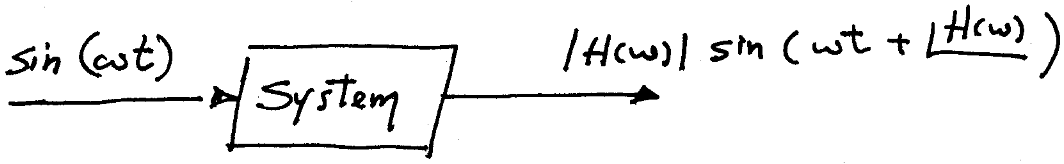
"the response of the system to an impulse $\delta(t)$" (2)

Note ~~the~~ an important difference:

(1) "response" is intended as multiplying a single input frequency (sinusoid) by $|H(\omega)|$ and phase shifting by $\angle H(\omega)$

(2) "response" is intended as the actual output we observe if we place an impulse $\delta(t)$ as input to the system.





$$\boxed{H(\omega) = \int_{-\infty}^{+\infty} h(t)\, e^{-j\omega t}\, dt}$$

What we do know from previous lectures:

$$x(t) \longrightarrow \boxed{H(\omega)} \longrightarrow y(t)$$

$$y(t) = FT^{-1}\left[X(j\omega)\cdot H(\omega) \right] \underset{\text{convolution property}}{=} \int_{-\infty}^{+\infty} x(\tau)\, h(t-\tau)\, d\tau = x \circledast h(t)$$

So we have an alternative way of computing the output of the system directly in time domain by solving the convolution integral of the input $x(t)$ with the impulse response $h(t)$

<u>INTUITIVE</u> <u>INTERPRETATION</u>

$$y(t) = FT^{-1}\left[X(j\omega)\cdot H(j\omega)\right] = \frac{1}{2\pi}\int_{-\infty}^{+\infty} X(j\omega)\, H(j\omega)\, e^{j\omega t}\, d\omega$$

$$= \frac{1}{2\pi}\int_{-\infty}^{+\infty} X(j\omega)\cdot |H(j\omega)|\, e^{j\omega t + \angle H(j\omega)}\, d\omega$$

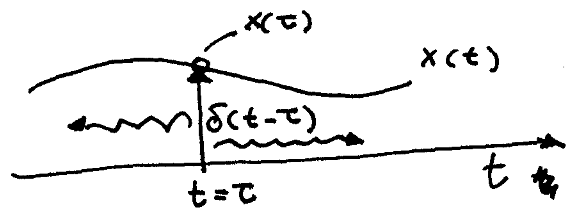
We have given an interpretation of this formula in previous lectures using the <u>superposition principle</u> : we are summing the responses at all frequencies ω, each weighted by the absolute value $|H(\omega)|$ and phase shifted by the phase $\angle H(\omega)$. Remember the correspondence with the Fourier Series in which the role of the coefficients c_k is replaced here by $X(j\omega)$.

In other words the signal $x(t)$ has a spectrum of frequencies $X(j\omega)$ and we are summing the single responses to each one of them using the integral representation.

Now, what is the <u>intuitive interpretation</u> of the convolution integral?

Remember that:

$$x(t) = \int_{-\infty}^{+\infty} x(\tau)\,\delta(t-\tau)\,d\tau$$



This says that $x(t)$ can be seen as a continuous sum of all of its values $x(\tau)$ obtained by "sweeping" the δ-function along the time-axis.

We have something similar by writing the output $y(t)$

$$y(t) = \int_{-\infty}^{+\infty} x(\tau)\,h(t-\tau)\,d\tau$$

Again we "sweep" along the time-axis but this time instead of the δ-function we use its response $h(t)$.

In other words, $x(t)$ is the ~~sum~~ (continuous) sum of δ-functions each weighted by the coefficient $x(\tau)$.
Similarly $y(t)$ is the (continuous) sum of the response of the δ-function, each weighted by the same coefficient $x(\tau)$.

Again we see a manifestation of the superposition principle!

Finally, let's prove the convolution property:

$$FT\left(\int_{-\infty}^{+\infty} x(\tau)h(t-\tau)\,d\tau\right) = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} x(\tau)\,h(t-\tau)\,e^{-j\omega t}\,d\tau\,dt$$

$$= \int_{-\infty}^{+\infty} x(\tau)\int_{-\infty}^{+\infty} h(t-\tau)\,e^{-j\omega t}\,dt\,d\tau \underset{\text{shifting property}}{=} \int_{-\infty}^{+\infty} x(\tau)\,e^{-j\omega\tau}\,H(j\omega)\,d\tau$$

$$= H(j\omega)\int_{-\infty}^{+\infty} x(\tau)\,e^{-j\omega\tau}\,d\tau = H(j\omega)\cdot X(j\omega)$$