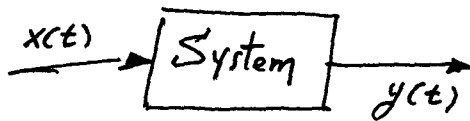


A little bit of review to make sure you have the big picture of what this class is teaching you.

We are learning the analysis of LINEAR SYSTEMS, for example RLC circuits.

The system can be viewed as a box transforming input in output.



① PHASOR METHOD

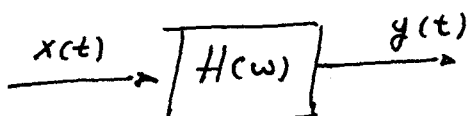
For system under sinusoidal excitation at a given frequency ω .

Idea since the linear system does not change the frequency, we can keep track of amplitude and phases only using phasors

$$\begin{array}{lcl}
 x(t) = A \cos(\omega t + \phi) & & Y = B e^{j\phi'} \\
 X = A e^{j\phi} & \longrightarrow \text{System} & y(t) = B \cos(\omega t + \phi')
 \end{array}$$

② FREQUENCY RESPONSE

The system can be described by the quantity $H(\omega)$ which is a complex number that describes the behavior of the system at any frequency ω . By using the freq. response we can find the output to a linear combination of signals at different frequencies:



Remember that $H(\omega)$ can be obtained as the output divided by input in complex phasor domain

$$H(\omega) = \frac{Y}{X}$$

$$x(t) = a_1 \sin(\omega_1 t + \phi_1) + a_2 \sin(\omega_2 t + \phi_2)$$

$$y(t) = a_1 |H(\omega_1)| \sin(\omega_1 t + \phi_1 + \angle H(\omega_1)) + a_2 |H(\omega_2)| \sin(\omega_2 t + \phi_2 + \angle H(\omega_2))$$

3

FOURIER SERIES

We can take The superposition principle and the freq. response method one step further. We have learned that any signal $x(t)$ that is

- periodic
- finite average power over one period

$$\frac{1}{T} \int_T |x(t)|^2 dt < \infty$$

can be represented as the sum of sinusoids:

$$x(t) = \sum_{k=-\infty}^{+\infty} C_k e^{jk\omega_0 t}$$

$$\omega_0 = \frac{2\pi}{T}$$

T fundamental period

$$C_k = \frac{1}{T} \int_T x(t) e^{-jk\omega_0 t} dt$$

It then follows that for linear systems:

$$x(t) \rightarrow \boxed{H(\omega)} \rightarrow y(t) = \sum_{k=-\infty}^{+\infty} C_k |H(\omega_k)| e^{j(k\omega_0 t + \angle H(\omega_k))}$$

In the output each harmonic component is multiplied by $|H(\omega_k)|$ and phase-shifted by $\angle H(\omega_k)$

4

FOURIER TRANSFORM

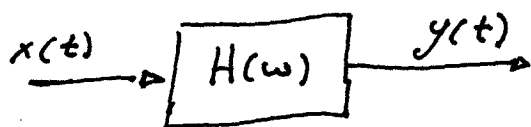
We take The Fourier series method one step further. Any signal $x(t)$ that has finite energy $\int_{-\infty}^{+\infty} |x(t)|^2 dt < \infty$

can be represented as a continuous sum (integral):

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} X(j\omega) e^{j\omega t} d\omega$$

$$X(j\omega) = \int_{-\infty}^{+\infty} x(t) e^{-j\omega t} dt$$

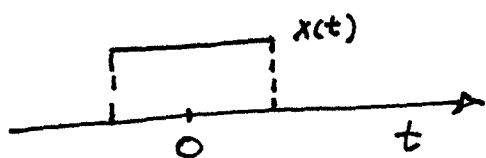
Notice the similarity between this representation and the Fourier Series. In fact we have derived in Lecture notes #6 this representation as the limit of the Fourier Series as the fundamental period of the signal $T \rightarrow \infty$



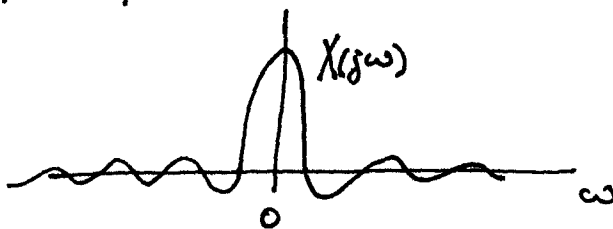
If we now pass the signal $x(t)$ through the system, we have a similar situation as before only that now we have a continuum SPECTRUM of frequencies rather than only multiples of ω_0 .

$$y(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\omega) H(\omega) e^{j\omega t} d\omega \quad (1)$$

We can still see this as the superposition of the effect of the system on all frequency components of the signal $x(t)$.



TIME DOMAIN



FREQUENCY DOMAIN

The system acts on all the frequency components of the spectrum $X(j\omega)$ of $x(t)$ via the function $H(\omega)$. This is the physical meaning of equation (1).

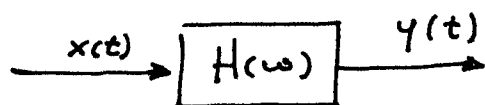
Remember also that the energy is conserved in the spectrum (PARSEVAL).
$$\int_{-\infty}^{\infty} |x(t)|^2 dt = \int_{-\infty}^{\infty} |X(j\omega)|^2 d\omega$$

in fact the FT is only a transform

$$x(t) \longleftrightarrow X(j\omega)$$

that gives a different representation of the same signal in different space. From time to frequency.

METHOD



- 1) Transform $x(t) \longleftrightarrow X(j\omega)$
- 2) Multiply $X(j\omega) \cdot H(\omega)$ to obtain $Y(j\omega)$
- 3) Inverse transform $Y(j\omega) \longleftrightarrow y(t)$

$$y(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} X(j\omega) e^{j\omega t} H(\omega) d\omega = FT^{-1}(X(j\omega) \cdot H(\omega))$$

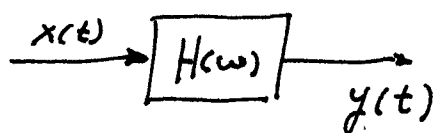
In last lecture LECTURE NOTES #8 we have introduced a special integral: the convolution integral.

Given two functions $f(t)$, $g(t)$

$$f \otimes g(t) = \int_{-\infty}^{+\infty} f(\tau) g(t - \tau) d\tau$$

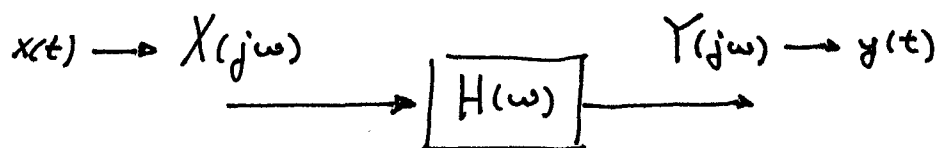
Then we mentioned an important property of Fourier Transforms: if we multiply two signals in the frequency domain, the inverse transform of the product corresponds to the convolution of the signals in time domain.

So if we define $h(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} H(\omega) e^{j\omega t} d\omega$
we have

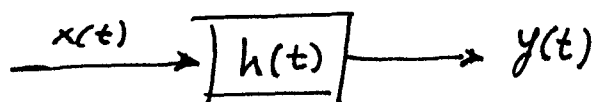


$$y(t) = FT^{-1}(X(j\omega) \cdot H(\omega)) = \int_{-\infty}^{+\infty} x(\tau) h(t - \tau) d\tau$$

So we have the "usual" method in frequency domain



Or a "new" method for solving the system in time domain



$$y(t) = x \otimes h(t)$$

The linear system can be represented by either $H(\omega)$ in freq. domain or by $h(t)$ in time domain. $h(t) \xleftrightarrow{FT} H(\omega)$

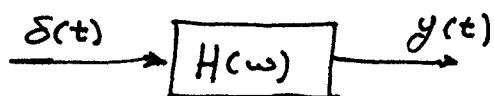
QUESTION

Can we see also here some PHYSICAL MEANING ?

Yes! Using the δ -function.

We have shown:

$$FT(\delta(t)) = \int_{-\infty}^{+\infty} \delta(t) e^{-j\omega t} dt = 1 \quad \text{so we have}$$



using the "usual" method $y(t) = FT^{-1}(1 \cdot H(\omega)) = h(t)$

So $h(t)$ is simply the output of the system to the δ -function that is the impulse response and $H(\omega)$ the Fourier Transform of the impulse response.

Moreover we have:

$$x(t) = \int_{-\infty}^{+\infty} x(\tau) \delta(t-\tau) d\tau$$

we called this in lecture notes #8 The convolution property of δ -function, it shows that a signal $x(t)$ can be represented as a continuous sum

by "sweeping" the δ -function along the time axis.

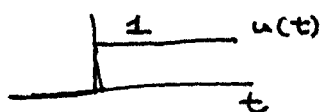
Now, using the "new" method we have

$$y(t) = \int_{-\infty}^{+\infty} x(\tau) h(t-\tau) d\tau$$

and we see another manifestation of the superposition principle. The output is a continuous sum of the responses to the δ -function, each weighted by the coefficient $x(\tau)$.

EXERCISE

FT of $u(t)$



This signal does not have finite energy. We need a trick similar to the one used for the constant signal: we use the δ -function

Let

$$f(t) = e^{-t/T} u(t)$$

and note that $\lim_{T \rightarrow \infty} f(t) = u(t)$

Also $f(t)$ has finite energy:

$$\begin{aligned} \int_{-\infty}^{+\infty} |f(t)|^2 dt &= \int_0^{+\infty} |e^{-t/T}|^2 dt \\ &= \int_0^{\infty} e^{-2t/T} dt \\ &= \frac{T}{2} \end{aligned}$$

$$\begin{aligned} F(j\omega) &= \int_0^{\infty} e^{-t/T} e^{-j\omega t} dt = \int_0^{\infty} e^{-(\frac{1}{T} + j\omega)t} dt \\ &= \left[-\frac{1}{\frac{1}{T} + j\omega} e^{-(\frac{1}{T} + j\omega)t} \right]_0^{\infty} \end{aligned}$$

$$= \frac{1}{\frac{1}{T} + j\omega} = \frac{1/T - j\omega}{\frac{1}{T^2} + \omega^2}$$

make sure you understand this equality

$$\lim_{T \rightarrow \infty} F(j\omega) = \underbrace{\lim_{T \rightarrow \infty} \frac{1/T}{1/T^2 + \omega^2}}_{\pi \delta(\omega)} - j \underbrace{\lim_{T \rightarrow \infty} \frac{\omega}{1/T^2 + \omega^2}}_{\frac{1}{\omega}}$$

Check:

$$\int_{-\infty}^{+\infty} \frac{1/T}{1/T^2 + \omega^2} d\omega = \pi \int_{-\infty}^{+\infty} \frac{1}{1 + T^2 \omega^2} d\omega = \int_{-\infty}^{+\infty} \frac{1}{1 + z^2} dz = \left[\arctan z \right]_{-\infty}^{+\infty} = \pi$$

$\omega T = z$
 $dz = T d\omega$

So the area of this function is constant and equal to π

However as $T \rightarrow \infty$ the function tends to zero anywhere except at $\omega = 0$ where it tends to ∞ .

So the whole function tends to $\pi \delta(\omega)$ and

$$u(t) \xleftrightarrow{FT} \pi \delta(\omega) + \frac{1}{j\omega}$$

Notice also:

$$1 = u(t) + u(-t)$$

$$FT(1) = FT(u(t)) + FT(u(-t))$$

$$= \pi \delta(\omega) + \cancel{\frac{1}{j\omega}} + \pi \delta(\omega) - \cancel{\frac{1}{j\omega}} = 2\pi \delta(\omega)$$

as expected!