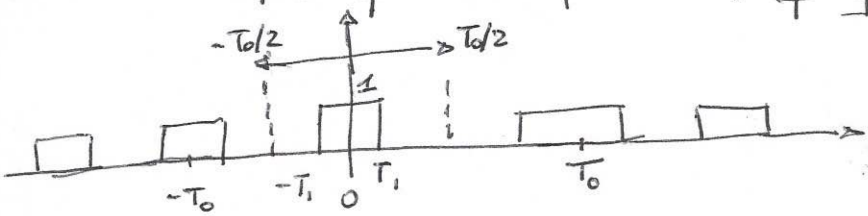
FOURIER SERIES PROBLEMS

Some examples of FS representations of periodic signals



$\omega_0 = \frac{2\pi}{T_0}$

Square wave of period T_0 and width $2T_1$

$$c_0 = \frac{1}{T_0}\int_{-T_0/2}^{T_0/2} f(t)\,dt = \frac{1}{T_0}\int_{-T_1}^{T_1} dt = \boxed{\frac{2T_1}{T_0}}$$

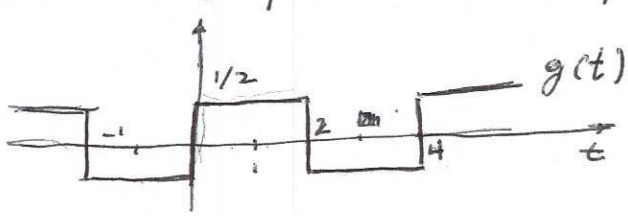
for $k \neq 0$ we have

$$c_k = \frac{1}{T_0}\int_{-T_1}^{T_1} e^{-jk\omega_0 t}\,dt = \frac{-1}{T_0\, jk\omega_0}\left[e^{-jk\omega_0 t}\right]_{-T_1}^{T_1}$$

$$= \frac{e^{jk\omega_0 T_1} - e^{-jk\omega_0 T_1}}{2j}\left(\frac{2}{T_0 k\omega_0}\right)$$

$$= \boxed{\frac{2}{T_0}\frac{\sin(k\omega_0 T_1)}{k\omega_0}}$$

Now consider a slightly modified version of the same signal



let's compute the coefficients from the previous result without computing integrals and using properties of F.S.

Square wave of period $T_0 = 4$ width $2T_1 = 2$

Compared to the previous signal, we can write:

$$\boxed{g(t) = f(t-1) - 1/2}$$

$2T_1 = 2 \quad T_0 = 4$

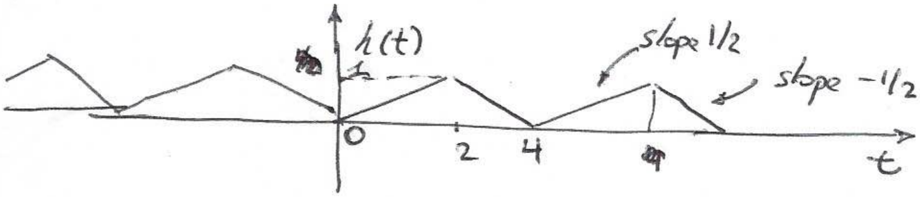
$T_1 = 1 \quad \omega_0 = \frac{2\pi}{T_0} = \frac{\pi}{2}$

So that we have:

$$c_0 = 1/2 - 1/2 = 0$$

$$\boxed{c_k = \frac{1}{\cancel{2}}\frac{\sin\left(\frac{\pi}{2}k\right)}{\frac{\pi k}{\cancel{2}}} e^{-j\pi k/2} = \frac{\sin(\pi/2\ k)}{k\pi} e^{-j\pi k/2}}$$

Now consider The Triangular wave



$T_0 = 4$
$\omega_0 = \frac{\pi}{2}$

Note that

$$\frac{d}{dt}\left[h(t) + \text{const}\right] = g(t)$$

so we can use the <u>derivative property</u> To find the Fourier coefficients

$$h(t) = \sum_k c_k e^{jk\omega_0 t} = \sum_{k \neq 0} c_k e^{jk\omega_0 t} + c_0$$

$$\frac{d[h(t) - c_0]}{dt} = \sum_{k \neq 0} \underbrace{jk\omega_0 c_k}_{\text{new coeff. } c_k'} e^{jk\omega_0 t}$$

$c_k' = jk\omega_0 c_k$ is the coeff. of $g(t)$ for $k \neq 0$

Therefore, The coeff. of $h(t)$ is $c_k = \frac{c_k'}{j\omega_0 k} = \frac{\sin(\pi/2\, k)}{k\pi\; j\omega_0 k} e^{-j\pi/2} = \boxed{\frac{2\sin((\pi/2)k)}{j(k\pi)^2} e^{-j\pi/2}}$ $\boxed{\text{for } k \neq 0}$

$$h(t) - c_0 = \sum_{k \neq 0} \frac{2\sin(\pi/2\, k)}{j(k\pi)^2} e^{-j\pi/2}$$

for $k = 0$, we have

$$c_0 = \frac{1}{T_0}\int_{T_0} h(t)\,dt = \frac{1}{2}$$

so, finally we have:

$$\boxed{h(t) = \frac{1}{2} + \sum_{k \neq 0} e^{jk\omega_0 t}\, 2\, \frac{\sin(k\pi/2)\, e^{-j\pi/2}}{j(k\pi)^2}}$$

Let $x_1(t)$ be periodic of fund. freq. ω_0

$$\boxed{x_1(t) = \sum_k c_k e^{jk\omega_0 t}}$$

Let $x_2(t) = x_1(1-t) + x_1(t-1)$

What is the fundamental freq. of $x_2(t)$?

Note that $x_1(t-1)$ is simply a shifted version of $x_1(t)$
similarly $x_1(1-t) = x_1(-(t-1))$ is a time reversed, version of the shifted signal.

Neither of these operations change the frequency
$\Rightarrow$ x_2 has fund. freq. ω_0

Find coeff. of $x_2(t)$ based on c_k

By the time-shift property:

$$x_1(t-1) = \sum c_k e^{-jk\omega_0} e^{jk\omega_0 t}$$

By the time-reversal property

(*) $$= x_1(-(t-1)) = \sum c_{-k} e^{-jk\omega_0} e^{jk\omega_0 t}$$

so that we have

$$x_1(t-1) + x_1(1-t) = \\ = \underbrace{(c_k + c_{-k}) e^{-jk\omega_0}}_{\text{This is the new coefficient}} e^{jk\omega_0 t}$$

Let's verify (*)

$$x(-t) = \sum_k c_k e^{-jk\omega_0 t}$$

$$x(-t+1) = \sum_k c_k e^{-jk\omega_0 t} e^{jk\omega_0}$$

$$= \sum_k c_k e^{jk\omega_0} e^{-jk\omega_0 t}$$

$$= \sum_k c_{-k} e^{-jk\omega_0} e^{+jk\omega_0 t}$$

Now let's consider a problem that uses properties of complex numbers -

Suppose That

1. $x(t)$ is real, $x(t) = \sum_k c_k e^{jk\omega_0 t}$
2. $x(t)$ has $T_0 = 4$
3. $c_k = 0$ for all $|k| > 1$
4. signal $y(t)$ with coeff $\left(b_k = e^{-j\pi \frac{k}{2}} c_{-k}\right)$ is odd
5. $\frac{1}{4}\int_0^4 |x(t)|^2 dt = \frac{1}{2}$

Find the possible expressions for $x(t)$

by (2) $\omega_0 = \pi/2$

by (3) $x(t) = c_0 + c_1 e^{j\frac{\pi}{2}t} + c_{-1} e^{-j\frac{\pi}{2}t}$

by (1) $c_k = c_{-k}^* \Rightarrow c_1 = c_{-1}^* \qquad c_1^* = c_{-1}$

so that, we have

$$x(t) = c_0 + c_1 e^{j\pi/2\, t} + c_1^* e^{-j\pi/2\, t}$$
$$= c_0 + c_1 e^{j\pi/2\, t} + \left(c_1 e^{j\pi/2\, t}\right)^*$$
$$= c_0 + 2\,\mathrm{Re}\left[c_1 e^{j\pi/2\, t}\right]$$

by (4) $y(t) = x(-(t-1)) = x(-t+1)$ [see previous problem derivation (*)]

now, $y(t)$ is odd & real $\Rightarrow$ coeff. of $y(t)$ are imaginary and $b_k = -b_{-k}$

so, we have $\boxed{b_1 = -b_{-1}}$ and $\boxed{b_0 = 0}$ because $b_0 = \frac{1}{T_0}\int_{T_0} y(t)\,dt$ would be real otherwise since $y(t)$ is real.

$$y(t) = b_1 e^{j\pi/2\, t} - b_1 e^{-j\pi/2\, t}$$

by (5) We have

$$\frac{1}{4}\int_0^4 |x(t)|^2 dt = \frac{1}{4}\int_0^4 |x(1-t)|^2 dt = \frac{1}{4}\int_0^4 |y(t)|^2 dt$$

$$\overset{\text{Parseval}}{=} |b_1|^2 + |b_{-1}|^2 = \frac{1}{2}$$

$$\Rightarrow |b_1|^2 = 1/4 \Rightarrow |b_1| = 1/2$$

It follows That since b_1 is purely imaginary it is $\boxed{b_1 = \pm j/2}$

CASE 1

$b_1 = j/2$

$$y(t) = x(1-t) = b_1 e^{j\pi/2 t} - b_1 e^{-j\pi/2 t} = j/2\, e^{j\pi/2 t} - j/2\, e^{-j\pi/2 t}$$

by (ii) $b_k = e^{-j\pi k/2} C_{-k} \Rightarrow$

$$C_0 = b_0 = 0$$

$$C_1 = b_{-1} e^{-j\pi k/2} = -j/2\, e^{-j\pi/2} = -1/2$$

$$x(t) = 2\,Re\left[-1/2\, e^{j\pi/2 t}\right] = \boxed{-\cos \pi/2\, t}$$

CASE 2

$b_1 = -j/2$

$$y(t) = x(1-t) = b_1 e^{j\pi/2 t} - b_1 e^{-j\pi/2 t} = -j/2\, e^{j\pi/2 t} + j/2\, e^{-j\pi/2 t}$$

$$C_0 = 0$$

$$C_1 e^{j\pi/2} = b_{-1}$$

$$C_1 = b_{-1} e^{-j\pi/2} = j/2\, e^{-j\pi/2} = 1/2$$

$$\boxed{x(t) = \cos \pi/2\, t}$$

Summary:

We have seen Fourier Series problems that use some of the properties of FS.
In general, to find the coefficients of the Fourier Series you need to solve a complex integral:

$$\left[\begin{array}{l} c_k = \frac{1}{T_0}\int_{T_0} x(t)\, e^{jk\omega_0 t}\, dt \\ \\ c_0 = \frac{1}{T_0}\int x(t)\, dt \end{array}\right.$$

However, using the properties many times you can avoid computing integral and re-use previous results.

- SHIFT
- TIME REVERSAL
- SCALED
- DC-OFFSET
- DERIVATIVE

Write the signal as a known signal on which some of these operations have been applied and proceed.

Other useful properties are
- PARSEVAL (average power is conserved in the coefficients)
- MULTPLICATION (coeff. of multiplication of two signals is given by convolution of coeff. of two signals.