An important FT. You find this FT in the Tables with $\frac{1}{T}=\alpha$

$$f(t) = u(t)\,e^{t/T} = u(t)\,e^{-\alpha t} \xleftrightarrow{FT} \boxed{\frac{1}{\alpha + j\omega}}$$

Q1 Can we compute the FT? Check the signal has finite energy

$$\int_{-\infty}^{+\infty} f(t)^2\,dt = \int_0^{\infty} e^{-\frac{2t}{T}}\,dt = \left[\frac{T e^{-2t/T}}{-2}\right]_0^{\infty} = \frac{T}{2}$$

Q2 Compute FT

$$F(j\omega) = \int_0^{\infty} e^{-t/T} e^{-j\omega t}\,dt = \int_0^{\infty} e^{-\left(\frac{1}{T} + j\omega\right)t}\,dt$$

$$= \left[\frac{-e^{-(1/T + j\omega)t}}{\frac{1}{T} + j\omega}\right]_0^{\infty} = \frac{1}{1/T + j\omega} = \frac{\frac{1}{T} - j\omega}{\frac{1}{T^2} + \omega^2} = \frac{\frac{1}{T}}{\frac{1}{T^2} + \omega^2} - j\,\frac{\omega}{\frac{1}{T^2} + \omega^2}$$

Q3 What happens for $T \to \infty$?

The energy increases and $f(t) \to u(t)$

What about $F(j\omega)$?

$$-j\,\frac{\omega}{\frac{1}{T^2} + \omega^2} \longrightarrow -j\,\frac{1}{\omega}$$

$$\frac{1/T}{\frac{1}{T^2} + \omega^2} \quad \begin{cases} \text{for } \omega = 0 & \text{it tends to } \infty \\ \text{for } \omega \neq 0 & \text{it tends to } 0 \end{cases}$$

so it looks like a spike in the origin.

Let's check the integral

$$\frac{1}{T}\int_{-\infty}^{+\infty} \frac{T^2}{1 + T^2\omega^2}\,d\omega = \int_{-\infty}^{+\infty} \frac{1}{1+z^2}\,dz = \left[\arctan z\right]_{-\infty}^{+\infty} = \pi$$

$T^2\omega^2 = z^2$

$dz = T\,d\omega$

So the second term tends to $\pi\delta(\omega)$ and we have

$$\boxed{FT(u(t)) = \pi\delta(\omega) + \frac{1}{j\omega}}$$

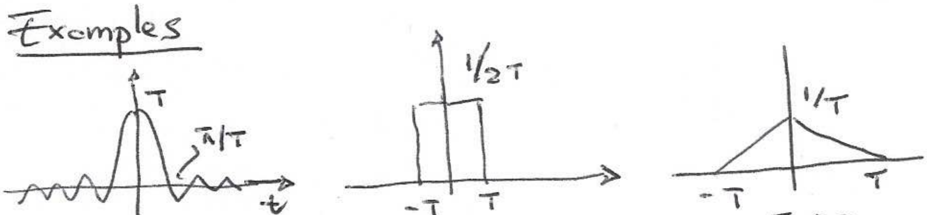
Note also that $u(t) + u(-t) = 1$ by the time scaling property, we have

$$FT(1) = \pi\delta(\omega) + \cancel{\frac{1}{j\omega}} + \pi\delta - \cancel{\frac{1}{j\omega}} = 2\pi\delta(\omega)$$

Consider now the derivative of the step function.

Some people define the δ-function as the derivative of the step.
We have defined the δ- as any function that peaks at origin and area is fixed.

Examples



$T \to \infty$ $\qquad$ $T \to 0$ $\qquad$ $T \to 0$

$\delta(t)$ $\qquad$ $\delta(t)$ $\qquad$ $\delta(t)$

We now compute the derivative of the step and check that it equals the δ-function.

$$\frac{d\,u(t)}{dt} = FT^{-1}\left[FT \frac{du(t)}{dt} \right]$$

$$\underset{\text{using derivative property}}{=} FT^{-1}\left[j\omega \left(\pi\delta(\omega) + \frac{1}{j\omega}\right)\right] = FT^{-1}\left(j\omega\pi\delta(\omega) + 1\right)$$

$$= \delta(t) + \frac{1}{2\pi}\underbrace{\cancel{\int_{-\infty}^{+\infty} j\omega\pi\delta(\omega)\, e^{j\omega t}\, d\omega}}_{0}$$

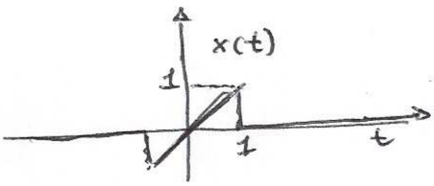
A general property of δ-function

$$\int_{-\infty}^{+\infty} f(t)\delta(t)\,dt = f(0)$$

So, we have

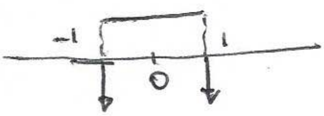
$$\boxed{\frac{d\,u(t)}{dt} = \delta(t)}$$

We can now apply the derivative of step-function as a property



In general the derivative of discontinuity is a δ-function of amplitude equal to the "jump" of the function

$$\frac{dx}{dt} = -\delta(t-1) + \text{Box}(2,0) - \delta(t+1)$$



Q Compute FT of x(t)

$$FT\left(\frac{dx}{dt}\right) = -e^{-j\omega} - e^{j\omega} + \frac{2\sin\omega}{\omega}$$

by shifting property of Transform of box

By integration property

$$FT(x(t)) = \frac{1}{j\omega}\left[-e^{-j\omega} - e^{j\omega} + \frac{2\sin\omega}{\omega}\right] + \cancel{\pi\, FT\left(\frac{dx}{dt}\right)\Big|_{\omega=0}}\ \delta(\omega)$$

0

$$= -\frac{e^{-j\omega} + e^{j\omega}}{j\omega} + \frac{2\sin\omega}{j\omega^2}$$

$$= -\frac{2\cos\omega}{j\omega} + \frac{2\sin\omega}{j\omega^2}$$

Consider The signal

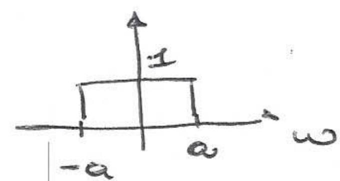
$$h(t) = \frac{\sin(at - 17a\pi^2)}{t - 17\pi^2}$$

Compute FT(h(t))

$$h(t) = \frac{\sin(a(t - 17\pi^2))}{t - 17\pi^2}$$

We know That (from Tables)

$$FT\left(\frac{\sin at}{\pi t}\right) \longleftrightarrow Box(1, 2a)$$



so that we also have by shifting property

$$\pi \frac{\sin(a(t - 17\pi^2))}{\pi(t - 17\pi^2)} \longleftrightarrow \pi\, Box(1, 2a)\, e^{-j17\pi^2\omega} = H(\omega)$$

Consider now LTI system with freq. response $H(\omega)$

$$x(t) \longrightarrow \boxed{H(\omega)} \xrightarrow{y(t)}$$

it is a pass-band with cut-off $\pm a$ of magnitude π over band & shift $-17\pi^2\omega$

Let $a = 2$

$x(t) = \cos(t)$

Compute Output

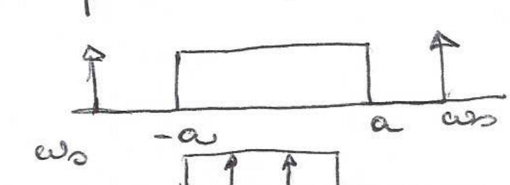
$y(t) = \pi \cos(t - 17\pi^2)$

Let $a = 2$

$x(t) = \cos(4t)$

$y(t) = 0$

Graphically we can represent $\cos(\omega_0 t) = \frac{e^{j\omega_0 t} + e^{-j\omega_0 t}}{2} \Rightarrow FT\cos(\omega_0 t) = \frac{\delta(\omega - \omega_0)}{2} + \frac{\delta(\omega + \omega_0)}{2}$



$\boxed{\omega_0 > a}$

$\boxed{\omega_0 < a}$