University of California, San Diego

ECE 45

Midterm - I

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Print your name: ____

Student ID number: ____

Note: No books, no notes, no calculators allowed.

Question	**Score**
1	
2	
3	
Total	

Some Useful Formulae:

$$Ae^{j\theta} = Acos(\theta) + jAsin(\theta)$$

$$cos(\theta) = \frac{e^{j\theta} + e^{-j\theta}}{2}, \qquad sin(\theta) = \frac{e^{j\theta} - e^{-j\theta}}{2j}$$

$$x(t) = \sum_{k=-\infty}^{+\infty} C_k e^{jk\omega_o t}, \qquad C_n = \frac{1}{T}\int_T x(t)\, e^{-jn\omega_o t} dt$$

Problem 1: [30 points]

Draw the Bode diagrams (magnitude and phase) for the following frequency response:

$$H(\omega) = 2 \cdot 10^3 \frac{0.5 + j\omega}{(j\omega)^2(5 + j\omega)(100 + j\omega)}$$

Please label all the critical points in your Bode diagrams.

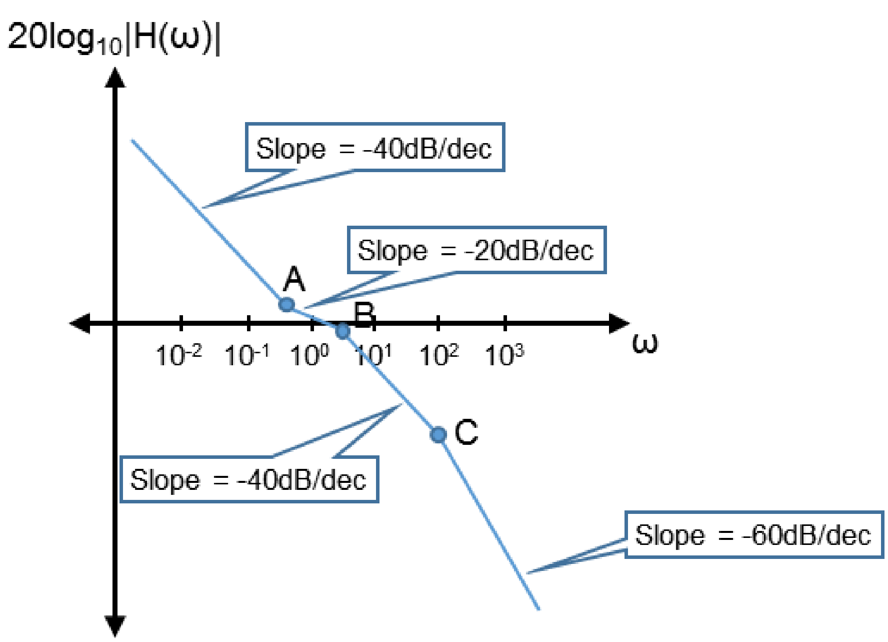
Solution:

First, we need to convert $H(\omega)$ into the normalized form.

$$H(\omega) = 2 \cdot 10^3 \frac{0.5\left(1 + \frac{j\omega}{0.5}\right)}{500(j\omega)^2\left(1 + \frac{j\omega}{5}\right)\left(1 + \frac{j\omega}{100}\right)} = 2\frac{\left(1 + \frac{j\omega}{0.5}\right)}{(j\omega)^2\left(1 + \frac{j\omega}{5}\right)\left(1 + \frac{j\omega}{100}\right)}$$

Now, we can draw the magnitude plot:

$$20\log_{10}|H(\omega)| = 20\log_{10} 2 + 20\log_{10}\left(\sqrt{1 + \left(\frac{\omega}{0.5}\right)^2}\right) - 40\log_{10}\omega + 20\log_{10}\left(\sqrt{1 + \left(\frac{\omega}{5}\right)^2}\right) - 20\log_{10}\left(\sqrt{1 + \left(\frac{\omega}{100}\right)^2}\right)$$



where,

For $\omega < 0.5$, the only active terms are $20\log_{10} 2 - 40\log_{10}\omega$.

So, at $\omega = 0.5$, height of A = $20\log_{10} 2 - 40\log_{10} 0.5 \cong 18dB$

From A to B, the slope is -20dB/decade. So, the drop in height from A to B is $20\log_{10}\frac{5}{0.5} = 20dB$

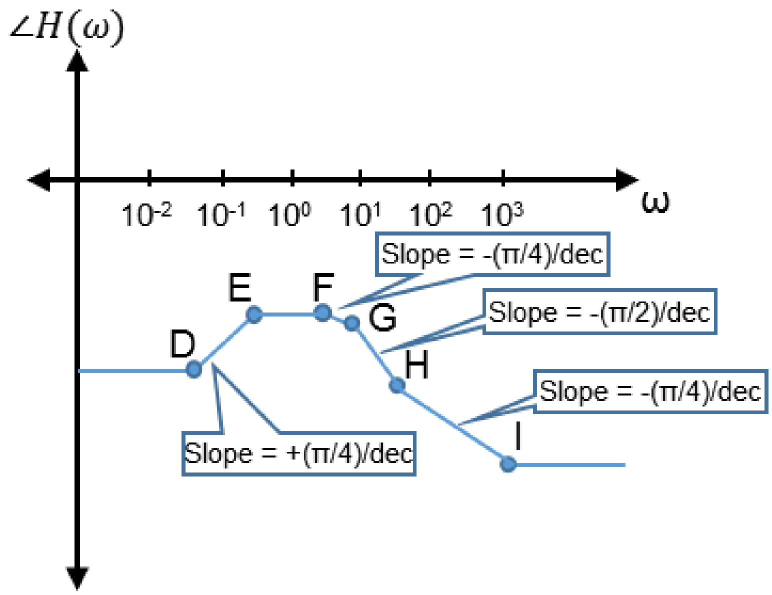
So, height of B = $18 - 20 = -2dB$

From B to C, the slope is -40dB/decade. So, the drop in height from B to C is $40\log_{10}\frac{100}{5} = 52dB$

So, height of C = $-2 - 52 = -54dB$

Now, we can draw the phase plot.

$$\angle H(\omega) = -\pi + \tan^{-1}\frac{\omega}{0.5} - \tan^{-1}\frac{\omega}{5} - \tan^{-1}\frac{\omega}{100}$$



where,

$D = (0.05, -\pi)$;

$E = \left(0.5, -\pi + \frac{\pi}{4} \cdot \log_{10}\frac{0.5}{0.05}\right) = \left(0.5, \frac{-3\pi}{4}\right)$

$F = \left(5, \frac{-3\pi}{4}\right)$

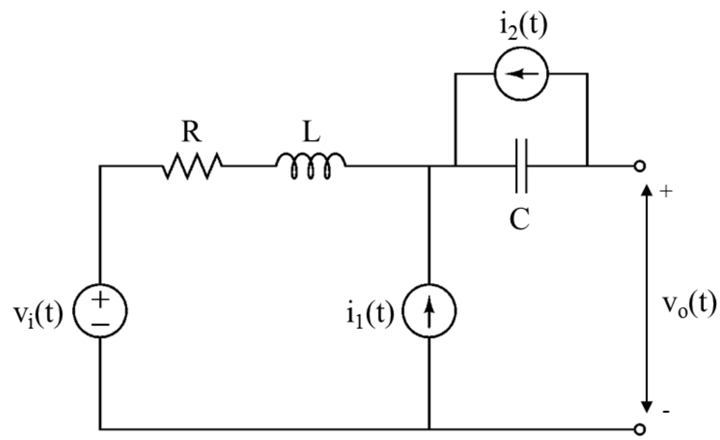
$G = \left(10, \frac{-3\pi}{4} - \frac{\pi}{4} \cdot \log_{10}\frac{10}{5}\right) = \left(10, \frac{-3\pi}{4} - \frac{\pi}{4} \cdot \log_{10} 2\right)$

$H = \left(50, \frac{-3\pi}{4} - \frac{\pi}{4} \cdot \log_{10}\frac{10}{5} - \frac{\pi}{2} \cdot \log_{10}\frac{50}{10}\right) = \left(50, \frac{-3\pi}{4} - \frac{\pi}{4} \cdot \log_{10} 50\right)$

$I = \left(1000, \frac{-3\pi}{4} - \frac{\pi}{4} \cdot \log_{10} 50 - \frac{\pi}{4} \cdot \log_{10}\frac{1000}{50}\right) = \left(1000, \frac{-3\pi}{2}\right)$

Problem 2: [Total: 40 points]

Consider the LTI system:



where, $v_i(t) = 2\cos(2t)$, $i_1(t) = \cos(2t + \pi/2)$, $i_2(t) = 2\cos(2t - \pi/2)$, R = 2, L = 1.

1. Find the value of C for which the Thevenin impedance, Z_{th}, is only resistor (i.e. Z_{th} is a purely real quantity) [15 points]

2. Now let C = 1/2 (not necessarily the correct answer for the above problem) and compute $v_o(t)$ [Hint: Superposition] [25 points]

Solution:

1) To find Z_{th}, short all voltage sources and open all current sources. Then,

$$Z_{th} = Z_R + Z_L + Z_C = R + j\omega L + \frac{1}{j\omega C} = R + j\left(\omega L - \frac{1}{\omega C}\right)$$

For Z_{th} to be purely real,

$$\left(\omega L - \frac{1}{\omega C}\right) = 0 \Rightarrow \omega^2 = \frac{1}{LC}$$

$$\therefore C = \frac{1}{\omega^2 L} \Rightarrow C = \frac{1}{4}$$

2) By superposition,

$$v_o(t) = v_{o1}(t) + v_{o2}(t) + v_{o3}(t)$$

where,

$v_{o1}(t)$ is the contribution of $v_{in}(t)$ to $v_o(t)$ when the other two sources are set to zero,

$v_{o2}(t)$ is the contribution of $i_1(t)$ to $v_o(t)$ when the other two sources are set to zero, and

$v_{o3}(t)$ is the contribution of $i_2(t)$ to $v_o(t)$ when the other two sources are set to zero

Using transfer functions,

$v_{o1} = v_i = 2e^{j0}$

Transfer function from $i_1(t)$ to $v_o(t)$ is: $R + j\omega L = 2 + j2$

$\therefore v_{o2} = i_1(2 + j2) = \left(e^{j\frac{\pi}{2}}\right)(2 + j2) = 2\sqrt{2}e^{j\left(\frac{\pi}{2}+\frac{\pi}{4}\right)} = 2\sqrt{2}e^{j\frac{3\pi}{4}}$

Transfer function from $i_2(t)$ to $v_o(t)$ is: $-\left(\frac{1}{j\omega C}\right) = j$

$\therefore v_{o2} = \left(2e^{-j\frac{\pi}{2}}\right)(j) = 2e^{j0}$

$\therefore v_o = 2e^{j0} + 2\sqrt{2}e^{j\frac{3\pi}{4}} + 2e^{j0} = 4 + 2\sqrt{2}\left(-\frac{1}{\sqrt{2}} + j\frac{1}{\sqrt{2}}\right) = 2 + j2$

$\therefore v_o(t) = 2\sqrt{2}cos\left(2t + \frac{\pi}{4}\right)$

Problem 3: [30 points]

A periodic function $f(t)$ with period, $T_0 = 6$, is defined on [-3, 3] as:

$$f(t) = \begin{cases} 0, & -3 \le t < -2 \\ 1, & -2 \le t < -1 \\ 0, & -1 \le t < 1 \\ -1, & 1 \le t < 2 \\ 0, & 2 \le t < 3 \end{cases}$$

Express $f(t)$ as a sum of complex exponentials, with each complex exponential term scaled by the appropriate Fourier series coefficient. Show all the steps leading to the expression of the coefficients.

[Extra credit: 5 points] If $g(t) = 2\,f\left(\frac{t}{3} - 1\right) + \frac{1}{2}$, find the average power of $g(t)$ i.e., $power\{g(t)\} = \frac{1}{T}\int_T \; g^2(t)\,dt$

Solution:

$f(t)$ can be represented as a sum of complex exponentials as

$$f(t) = \sum_{n=-\infty}^{+\infty} C_n e^{jn\omega_o t}, \qquad where\ C_n = \frac{1}{T}\int_T \; f(t)\, e^{-jn\omega_o t} dt$$

Here, the period $T_o = 6 \Rightarrow \omega = \frac{\pi}{3}$

So,

When $n = 0$,

$$C_0 = \frac{1}{6}\int_{-3}^{+3} f(t)\,dt = \frac{1}{6}\int_{-2}^{-1} 1\,dt + \frac{1}{6}\int_{+1}^{+2} (-1)\,dt = 0$$

When $n \neq 0$,

$$C_n = \frac{1}{6}\int_{-3}^{+3} f(t)\, e^{-jn\omega_o t} dt$$

$$\therefore C_n = \frac{1}{6}\left[\int_{-2}^{-1} 1\, e^{-jn\omega_o t} dt + \int_{+1}^{+2} (-1)\, e^{-jn\omega_o t} dt\right]$$

$$\therefore C_n = \frac{1}{6}\left[\frac{1}{-jn\omega_o} e^{-jn\omega_o t}\right]_{-2}^{-1} - \frac{1}{6}\left[\frac{1}{-jn\omega_o} e^{-jn\omega_o t}\right]_{+1}^{+2}$$

$$\therefore C_n = \frac{1}{6}\left[\frac{1}{-jn\omega_o}\left(e^{jn\omega_o 1} - e^{jn\omega_o 2}\right)\right] \quad - \frac{1}{6}\left[\frac{1}{-jn\omega_o}\left(e^{-jn\omega_o 2} - e^{-jn\omega_o 1}\right)\right]$$

$$\therefore C_n = \frac{1}{j6n\omega_o}\left[\left(e^{jn\omega_o 2} + e^{-jn\omega_o 2}\right) - \left(e^{jn\omega_o 1} + e^{-jn\omega_o 1}\right)\right]$$

$$\therefore C_n = \frac{1}{j3n\omega_o}\left[\frac{\left(e^{jn\omega_o 2} + e^{-jn\omega_o 2}\right)}{2} - \frac{\left(e^{jn\omega_o 1} + e^{-jn\omega_o 1}\right)}{2}\right]$$

$$\therefore C_n = \frac{1}{j3n\omega_o}[\cos(2n\omega_o) - \cos(n\omega_o)]$$

Extra credit solution:

Since, $g(t) = 2\,f\left(\frac{t}{3} - 1\right) + \frac{1}{2}$, period of g(t) = 3T$_0$ = 18. Using definition of g(t), we can now find it's value within one period [-8, +10]:

$$g(t) = \begin{cases} \frac{1}{2}, & -8 \le t < -5 \\ \frac{5}{2}, & -5 \le t < -2 \\ \frac{1}{2}, & -2 \le t < 4 \\ -\frac{3}{2}, & 4 \le t < 7 \\ \frac{1}{2}, & 7 \le t < 10 \end{cases}$$

$$power\{g(t)\} = \frac{1}{T}\int_T g^2(t)\,dt = \frac{1}{18}\int_{-8}^{+10} g^2(t)\,dt$$

$$= \frac{1}{18}\left[\int_{-8}^{-5}\frac{1}{4}dt + \int_{-5}^{-2}\frac{25}{4}dt + \int_{-2}^{+4}\frac{1}{4}dt + \int_{+4}^{+7}\frac{9}{4}dt + \int_{+7}^{+10}\frac{1}{4}dt\right]$$

$$= \frac{1}{18}\left[\frac{3}{4} + \frac{75}{4} + \frac{6}{4} + \frac{27}{4} + \frac{3}{4}\right]$$

$$= \frac{1}{18}\left[\frac{114}{4}\right] = \frac{19}{12}$$