University of California, San Diego
ECE 45 Spring 2019
MIDTERM EXAM 1

Massimo Franceschetti

Print your name: Joseph Fourier

Student ID Number: $F_n = \frac{1}{T}\int_{t_0}^{t_0+T} f(t)\, e^{-jn\omega_0 t}\, dt$

Note: No books, calculators, or other electronic devices allowed.

Question	Score
Problem 1	10 /10
Problem 2	15 /15
Problem 3	15 /15

Useful Formulas:

Some Useful Formulas:

$$Ae^{j\theta} = A\cos(\theta) + jA\sin(\theta)$$

$$\cos(\theta) = \frac{e^{j\theta} + e^{-j\theta}}{2} \quad , \quad \sin(\theta) = \frac{e^{j\theta} - e^{-j\theta}}{2j}$$

$$x(t) = \sum_{k=-\infty}^{\infty} C_k e^{jk\omega_0 t} \quad , \quad C_n = \frac{1}{T}\int_T x(t) e^{-jn\omega_0 t} dt$$

206 Fourier Series Representation of Periodic Signals Chap. 3

TABLE 3.1 PROPERTIES OF CONTINUOUS-TIME FOURIER SERIES

Property	Section	Periodic Signal	Fourier Series Coefficients
		$x(t)$, $y(t)$: Periodic with period T and fundamental frequency $\omega_0 = 2\pi/T$	a_k, b_k
Linearity	3.5.1	$Ax(t) + By(t)$	$Aa_k + Bb_k$
Time Shifting	3.5.2	$x(t - t_0)$	$a_k e^{-jk\omega_0 t_0} = a_k e^{-jk(2\pi/T)t_0}$
Frequency Shifting		$e^{jM\omega_0 t} = e^{jM(2\pi/T)t} x(t)$	a_{k-M}
Conjugation	3.5.6	$x^*(t)$	a^*_{-k}
Time Reversal	3.5.3	$x(-t)$	a_{-k}
Time Scaling	3.5.4	$x(\alpha t)$, $\alpha > 0$ (periodic with period T/α)	a_k
Periodic Convolution		$\int_T x(\tau) y(t-\tau) d\tau$	$Ta_k b_k$
Multiplication	3.5.5	$x(t)y(t)$	$\sum_{l=-\infty}^{\infty} a_l b_{k-l}$
Differentiation		$\frac{dx(t)}{dt}$	$jk\omega_0 a_k = jk\frac{2\pi}{T} a_k$
Integration		$\int_{-\infty}^{t} x(t) dt$ (finite valued and periodic only if $a_0 = 0$)	$\left(\frac{1}{jk\omega_0}\right) a_k = \left(\frac{1}{jk(2\pi/T)}\right) a_k$
Conjugate Symmetry for Real Signals	3.5.6	$x(t)$ real	$\begin{cases} a_k = a^*_{-k} \\ \Re e\{a_k\} = \Re e\{a_{-k}\} \\ \Im m\{a_k\} = -\Im m\{a_{-k}\} \\ \lvert a_k \rvert = \lvert a_{-k} \rvert \\ \sphericalangle a_k = -\sphericalangle a_{-k} \end{cases}$
Real and Even Signals	3.5.6	$x(t)$ real and even	a_k real and even
Real and Odd Signals	3.5.6	$x(t)$ real and odd	a_k purely imaginary and odd
Even-Odd Decomposition of Real Signals		$x_e(t) = \mathcal{E}v\{x(t)\}$ [$x(t)$ real]	$\Re e\{a_k\}$
		$x_o(t) = \mathcal{O}d\{x(t)\}$ [$x(t)$ real]	$j\Im m\{a_k\}$

Parseval's Relation for Periodic Signals

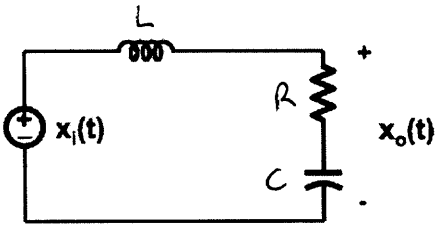
$$\frac{1}{T}\int_T |x(t)|^2 dt = \sum_{k=-\infty}^{\infty} |a_k|^2$$

Bode Plot Approximations:

$$\log_{10}\left(\sqrt{1 + \left(\frac{x}{a}\right)^2}\right) \approx \begin{cases} \log_{10}(x/a) & x \ge a \\ 0 & x < a \end{cases}$$

$$\tan^{-1}\left(\frac{x}{a}\right) \approx \begin{cases} 0 & x < \frac{a}{10} \\ \pi/4 \log_{10}\left(\frac{10x}{a}\right) & \frac{a}{10} \le x < 10a \\ \pi/2 & x \ge 10a \end{cases}$$

Problem 1:



a) Derive the frequency response H(w) of the system above. (5 points)
b) What is $x_o(t)$ in response to $x_i(t) = 3 + cos(2t)$? Assume L = ¼, R = C = 1 (5 points)

a)

$$X_o = \frac{Z_C + Z_R}{Z_C + Z_R + Z_L} X_i$$

$$X_o = \frac{1/j\omega C + R}{1/j\omega C + R + j\omega L} X_i$$

$$X_o = \frac{1 + j\omega RC}{1 + j\omega RC + j^2\omega^2 LC} X_i$$

$$\frac{X_o}{X_i} = H(\omega) = \underline{\frac{1 + j\omega RC}{1 - \omega^2 LC + j\omega RC}}$$

b)

$$H(0) = \frac{1 + j\cdot 0}{1 - 0 + j\cdot 0} = 1$$

$$H(2) = \frac{1 + j2\cdot 1\cdot 1}{1 - 2^2\cdot 1/4\cdot 1 + j\cdot 2\cdot 1\cdot 1}$$

$$= \frac{1 + j2}{j2}$$

$$= \frac{1}{j2} + 1$$

$$H(2) = 1 - 1/2\, j = \sqrt{1 + (1/2)^2}\, e^{j\tan^{-1}(-1/2)}$$

$$H(2) = \sqrt{5}/2\, e^{j\tan^{-1}(-1/2)}$$

$$x_o(t) = 3\cdot H(0) + \cos(2t)\cdot H(2)$$

$$\underline{x_o(t) = 3 + \frac{\sqrt{5}}{2}\cos(2t + \tan^{-1}(-1/2))}$$

Problem 2:

Graph the magnitude and phase of the following frequency response on a Bode Plot (12 points):

$$H(w) = \frac{100*jw*(10^6 + jw)}{[(jw)^2 + (jw)^3*10^{-3}]*[10^5+jw]}$$

What type of filter is it? (ex: lowpass, highpass, bandpass, ...) (3 points)

2. $$H(\omega) = \frac{100 * j\omega * (10^6 + j\omega)}{[(j\omega)^2 + (j\omega)^3 10^{-3}] * [10^5 + j\omega]}$$

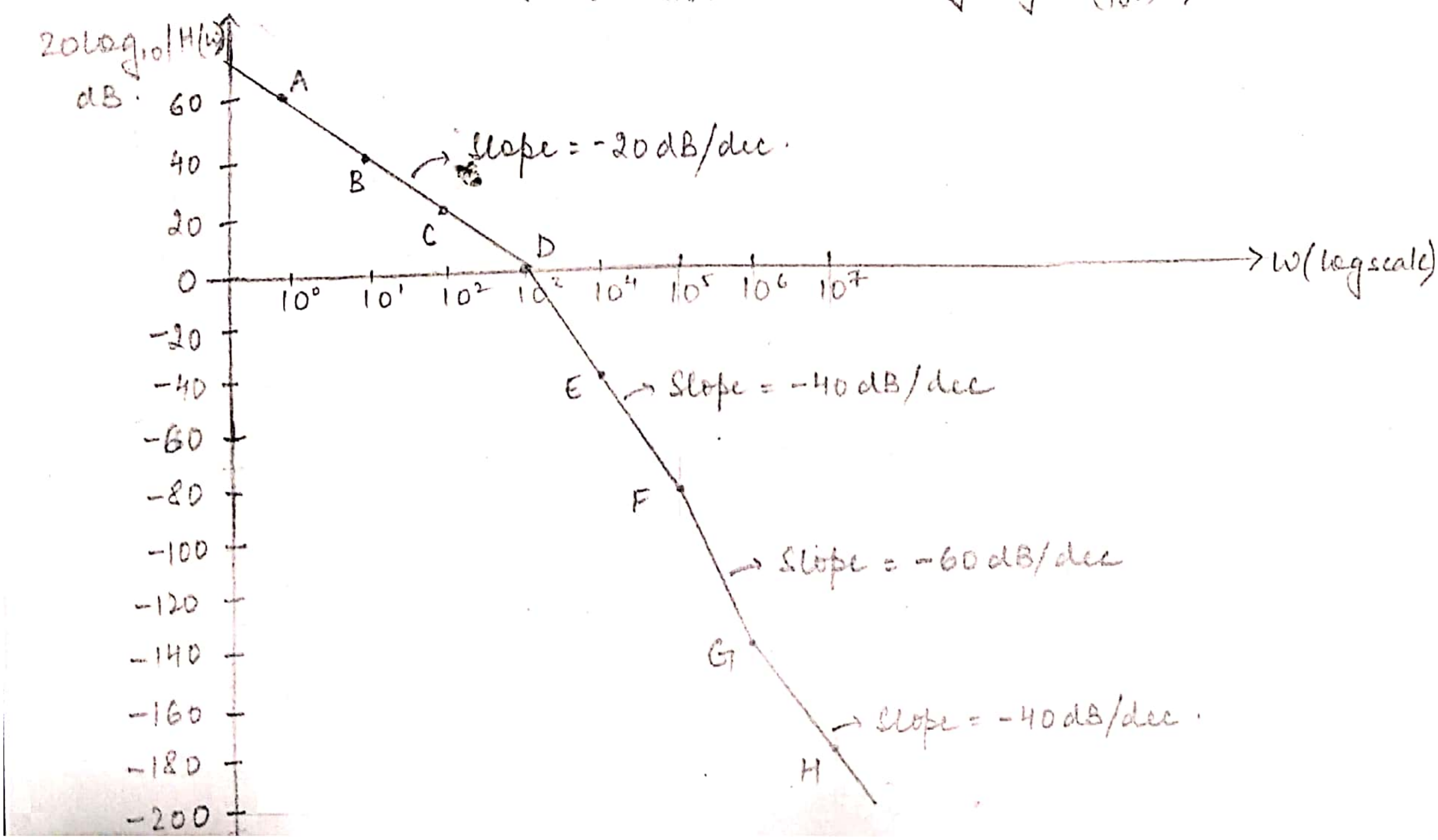
Writing in the standard form,

$$H(\omega) = \frac{100 \cdot (j\omega) \cdot 10^6 \left(1 + \frac{j\omega}{10^6}\right)}{(j\omega)^2 \left(1 + \frac{j\omega}{10^3}\right) \cdot 10^5 \left(1 + \frac{j\omega}{10^5}\right)}$$

$$= \frac{1000\,(j\omega)\left(1 + \frac{j\omega}{10^6}\right)}{(j\omega)^2 \cdot \left(1 + \frac{j\omega}{10^3}\right)\left(1 + \frac{j\omega}{10^5}\right)} = \frac{1000 \left(1 + \frac{j\omega}{10^6}\right)}{(j\omega)\left(1 + \frac{j\omega}{10^3}\right)\left(1 + \frac{j\omega}{10^5}\right)}$$

Drawing the magnitude plot:

$$20\log_{10}|H(\omega)| = 20\log_{10}(1000) + 20\log_{10}\left(\sqrt{1 + \left(\frac{\omega}{10^6}\right)^2}\right) - 20\log_{10}\omega - 20\log_{10}\left(\sqrt{1 + \left(\frac{\omega}{10^3}\right)^2}\right) - 20\log_{10}\left(\sqrt{1 + \left(\frac{\omega}{10^5}\right)^2}\right)$$



A : $(\omega, dB) = (1, 20\log(1000) - 20\log 1)$

B $= (10, 20\log 1000 - 20\log 10)$

C $= (10^2, 20\log 1000 - 20\log 100)$

D $= (10^3, 20\log 1000 - 20\log 1000 - 20\log(10^3/10^2))$

E $= (10^4, 20\log 1000 - 20\log 10^4 - 20\log(10^4/10^2))$

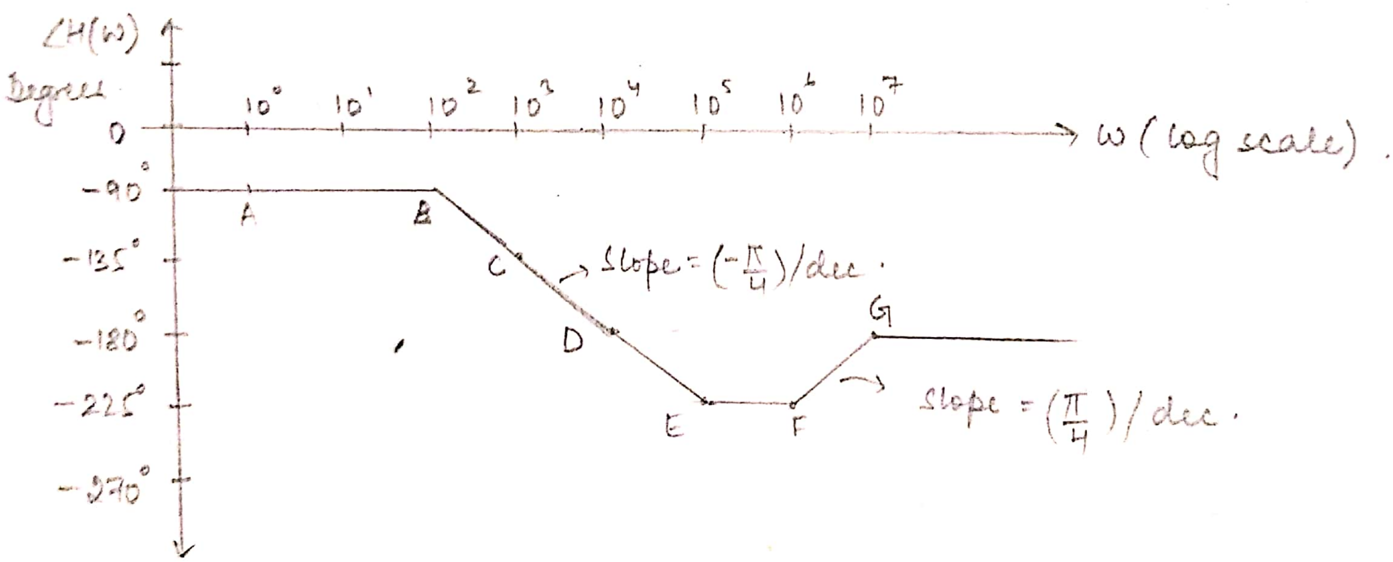
F $= (10^5, 20\log 10^3 - 20\log 10^5 - 20\log(10^5/10^2) - 20\log(10^5/10^5))$

G $= (10^6, 20\log 10^3 - 20\log 10^6 - 20\log(10^6/10^2) - 20\log(10^6/10^5)$
$+ 20\log(10^6/10^6))$

H $= (10^7, 20\log 10^3 - 20\log 10^7 - 20\log(10^7/10^2) - 20\log(10^7/10^5)$
$+ 20\log(10^7/10^6))$

Drawing the phase plot :

$$\angle H(\omega) = -\frac{\pi}{2} + \tan^{-1}\left(\frac{\omega}{10^6}\right) - \tan^{-1}\left(\frac{\omega}{10^2}\right) - \tan^{-1}\left(\frac{\omega}{10^5}\right)$$



A $= (\omega, \theta) = (1, -90^\circ)$

B $= (100, -90^\circ - \tan^{-1}(100/10^2))$

C $= (1000, -90^\circ - \tan^{-1}(10^3/10^2))$

D $= (10^4, -90^\circ - \tan^{-1}(10^4/10^2) - \tan^{-1}(10^4/10^5))$

E $= (10^5, -180^\circ - \tan^{-1}(10^5/10^5) + \tan^{-1}(10^5/10^6))$

F $= (10^6, -180^\circ - \tan^{-1}(10^6/10^5) + \tan^{-1}(10^6/10^6))$

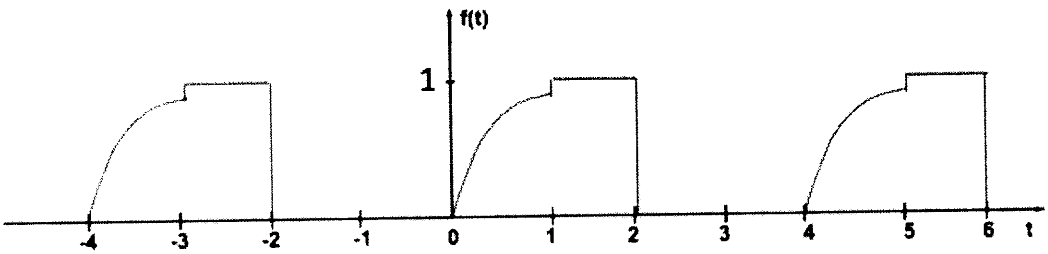
G $= (10^7, -270 + \tan^{-1}(10^7/10^6))$.

2. From the bode plot, we can see that it is a low pass filter (gain reduces on increasing the frequency)

Problem 3:

a) Find the fourier series coefficients of the following periodic function with period T = 4. An approximate graph of this function can be seen below for your convenience. You MUST use the integration formula to calculate the fourier series coefficients. (10 points)

$$f(t) = \begin{cases} 1-e^{-t} & 0 \le t < 1 \\ 1 & 1 \le t \le 2 \end{cases}$$



$$F_n = \frac{1}{T}\int_T f(t)\, e^{-jn\omega_0 t}\, dt \qquad \omega_0 = \frac{2\pi}{T} = \frac{2\pi}{4} = \frac{\pi}{2}$$

$$F_n = \frac{1}{4}\int_0^4 f(t)\, e^{-jn\omega_0 t}\, dt$$

$$F_n = \frac{1}{4}\left[\int_0^1 (1-e^{-t})\, e^{-jn\omega_0 t}\, dt + \int_1^2 1\cdot e^{-jn\omega_0 t}\, dt\right]$$

$$F_n = \frac{1}{4}\left[\int_0^1 e^{-jn\omega_0 t} - e^{-(1+jn\omega_0)t}\, dt + \int_1^2 e^{-jn\omega_0 t}\, dt\right]$$

$$F_n = \frac{1}{4}\left[\left(\frac{1}{-jn\omega_0} e^{-jn\omega_0 t} - \frac{1}{-(1+jn\omega_0)} e^{-(1+jn\omega_0)t}\right)\Big|_0^1 + \left(\frac{1}{-jn\omega_0} e^{-jn\omega_0 t}\right)\Big|_1^2\right]$$

$$F_n = \frac{1}{4}\left[\left(\frac{1}{-jn\omega_0} e^{-jn\omega_0} + \frac{1}{1+jn\omega_0} e^{-(1+jn\omega_0)} - \frac{1}{-jn\omega_0} + \frac{1}{-(1+jn\omega_0)}\right) + \left(\frac{1}{-jn\omega_0} e^{-jn\omega_0 2} - \frac{1}{-jn\omega_0} e^{-jn\omega_0}\right)\right]$$

$$F_n = \frac{1}{4}\left[\left(\frac{1}{-jn\omega_0} e^{-jn\omega_0} + \frac{1}{1+jn\omega_0} e^{-1} e^{-jn\omega_0} + \frac{1}{jn\omega_0} - \frac{1}{1+jn\omega_0}\right) + \left(\frac{1}{-jn\omega_0} e^{-jn\omega_0 2} + \frac{1}{jn\omega_0} e^{-jn\omega_0}\right)\right]$$

$$\omega_0 = \frac{\pi}{2} \qquad e^{-jn\omega_0} = (e^{-j\pi/2})^n = (-j)^n, \qquad e^{-jn\omega_0 2} = (e^{-j\pi/2\cdot 2})^n = (e^{-j\pi})^n = (-1)^n$$

$$F_n = \frac{1}{4}\left[\left(\frac{-1}{jn\pi/2}(-j)^n + \frac{1}{1+jn\pi/2} e^{-1}(-j)^n + \frac{1}{jn\pi/2} - \frac{1}{1+jn\pi/2}\right) + \left(\frac{-1}{jn\pi/2}(-1)^n + \frac{1}{jn\pi/2}(-j)^n\right)\right]$$

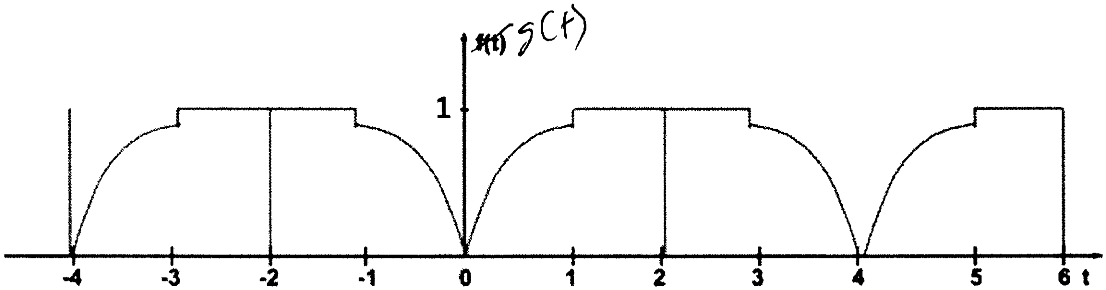
$$F_n = \frac{1}{4}\left[\frac{1}{jn\pi/2}\left(-(-j)^n + 1 - (-1)^n + (-j)^n\right) + \frac{1}{1+jn\pi/2}\left(e^{-1}(-j)^n - 1\right)\right]$$

$$F_n = \frac{1}{4}\left[\frac{1}{jn\pi/2}\left(1 - (-1)^n\right) + \frac{1}{1+jn\pi/2}\left(e^{-1}(-j)^n - 1\right)\right]$$

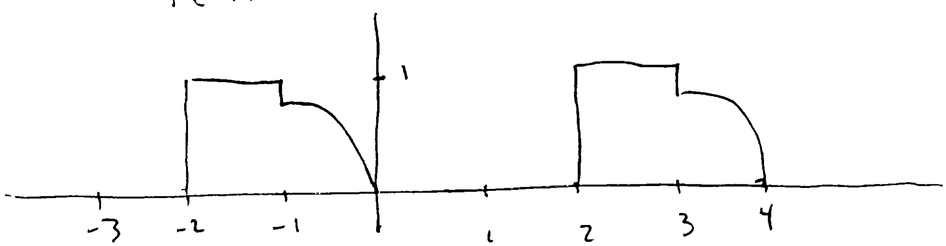
$$F_0 = \frac{1}{4}\int_0^1 1 - e^{-t}\, dt + \frac{1}{4}\int_1^2 1\, dt = \frac{1}{4}\left(t + e^{-t}\right)\Big|_0^1 + \frac{1}{4}\, t\,\Big|_1^2 = \frac{1}{4}\left(1 + e^{-1} - 1\right) + \frac{1}{4}(2-1)$$

$$\underline{F_0 = \tfrac{1}{4} e^{-1} + \tfrac{1}{4}}$$

b) Using the fourier series coefficients found above and fourier series properties from class, find the fourier series coefficients of the function below. (5 points)



f(-t)



$g(t) = f(t) + f(-t)$

$\underline{G_n = F_n + F_{-n}}$