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ECE 45 Homework 4 Solutions

Problem 4.1

Simplify the following expressions as much as possible:

(a) $a(t) = (1+t^2)\,(\delta(t) - 2\delta(t-2))$

(b) $b(t) = \cos(2\pi t)\,\left(\frac{du(t)}{dt} + \delta(t+1/4)\right)$

(c) $c(t) = \sin(2\pi t)\,\delta(1/2 - 2t)$

(d) $d(t) = \int_{-6}^{\infty}(\tau^2+6)\,\delta(\tau-2)\,d\tau$

(e) $e(t) = \int_{6}^{\infty}(\tau^2+6)\,\delta(\tau-2)\,d\tau$

(f) $f(t) = \int_{-\infty}^{t}\delta(\tau-2)\,d\tau$

(g) $g(t) = u(t) * (\delta(t+2) - \delta(t-2))$

Solutions

We use the following properties of the impulse function:

$$x(t)\,\delta(t-t_0) = x(t_0)\,\delta(t-t_0)$$
$$\delta(at) = \frac{1}{|a|}\,\delta(t)$$
$$\int_a^b \delta(\tau)\,d\tau = \begin{cases} 1 & \text{if } a < 0 < b \\ 0 & \text{otherwise.} \end{cases}$$
$$\frac{d}{dt}u(t) = \delta(t)$$
$$x(t) * \delta(t-t_0) = x(t-t_0).$$

(a) $a(t) = \delta(t) - 2\delta(t-2) + t^2\delta(t) - 2t^2\delta(t-2) = \delta(t) - 10\delta(t-2)$

(b) $b(t) = \cos(2\pi t)\,\delta(t) + \cos(2\pi t)\delta(t+1/4) = \delta(t)$

(c) $c(t) = \sin(2\pi t)\,\delta(-2(t-1/4)) = \frac{1}{2}\sin(2\pi t)\,\delta(t-1/4) = \frac{1}{2}\,\delta(t-1/4)$

(d) $d(t) = 10\int_{-6}^{\infty}\delta(\tau-2)\,d\tau = 10$

(e) $e(t) = 10\int_{6}^{\infty}\delta(\tau-2)\,d\tau = 0$

(f) $f(t) = \int_{-\infty}^{t}\delta(\tau-2)\,d\tau = \begin{cases} 1 & \text{if } t \geq 2 \\ 0 & \text{otherwise} \end{cases} = u(t-2)$

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(g) $g(t) = u(t) * \delta(t+2) - u(t) * \delta(t-2) = u(t+2) - u(t-2) = \begin{cases} 1 & \text{if } -2 \le t < 2 \\ 0 & \text{otherwise.} \end{cases}$

Problem 4.2

Let $f(t) = \begin{cases} 2 - |t| & \text{if } |t| \le 2 \\ 0 & \text{otherwise.} \end{cases}$

(a) Determine the function $y(t) = f(t) * u(t)$.

(b) Determine the function $z(t) = \frac{df(t)}{dt} * u(t)$.

(c) Determine the function $w(t) = f(t) * \frac{du(t)}{dt}$.

Solutions

We first note that

$$f(t) = \begin{cases} 2 + t & \text{if } -2 \le t < 0 \\ 2 - t & \text{if } 0 \le t \le 2 \\ 0 & \text{otherwise.} \end{cases} = 2\Delta\left(\frac{t}{4}\right)$$

(a) We have $u(t-\tau) = 1$ if $\tau \le t$ and $u(t-\tau) = 0$ if $\tau > t$, so

$$y(t) = \int_{-\infty}^{\infty} u(t-\tau)\, f(\tau)\, d\tau = \int_{-\infty}^{t} f(\tau)\, d\tau = \begin{cases} 0 & \text{if } t < -2 \\ \int_{-2}^{t} (2+\tau)\, d\tau & \text{if } -2 \le t < 0 \\ \int_{-2}^{0} (2+\tau)\, d\tau + \int_{0}^{t} (2-\tau)\, d\tau & \text{if } 0 \le t < 2 \\ \int_{-2}^{0} (2+\tau)\, d\tau + \int_{0}^{2} (2-\tau)\, d\tau & \text{if } t \ge 2 \end{cases}$$

$$= \begin{cases} 0 & \text{if } t < -2 \\ \dfrac{(t+2)^2}{2} & \text{if } -2 \le t < 0 \\ \dfrac{4 + 4t - t^2}{2} & \text{if } 0 \le t < 2 \\ 4 & \text{if } t \ge 2. \end{cases}$$

(b) We have $\frac{df(t)}{dt} = \begin{cases} 1 & \text{if } -2 \le t < 0 \\ -1 & \text{if } 0 \le t \le 2 \\ 0 & \text{otherwise.} \end{cases}$

$$z(t) = \int_{-\infty}^{\infty} u(t-\tau)\,\frac{df(\tau)}{d\tau}\,d\tau = \int_{-\infty}^{t} \frac{df(\tau)}{d\tau}\,d\tau = \begin{cases} 0 & \text{if } t < -2 \\ \int_{-2}^{t} 1\,d\tau & \text{if } -2 \le t < 0 \\ \int_{-2}^{0} 1\,d\tau + \int_{0}^{t} (-1)\,d\tau & \text{if } 0 \le t < 2 \\ \int_{-2}^{0} 1\,d\tau + \int_{0}^{2} (-1)\,d\tau & \text{if } t \ge 2 \end{cases}$$

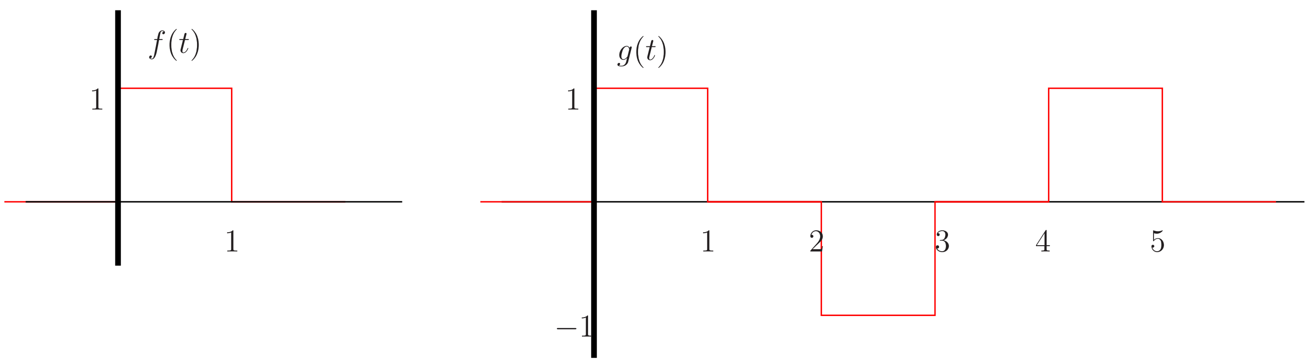
$$= \begin{cases} 0 & \text{if } t < -2 \\ 2+t & \text{if } -2 \le t < 0 \\ 2-t & \text{if } 0 \le t < 2 \\ 0 & \text{if } t \ge 2 \end{cases} \quad = f(t).$$

(c) We have $\frac{du(t)}{dt} = \delta(t)$, so

$$w(t) = \frac{du(t)}{dt} * f(t) = \delta(t) * f(t) = f(t).$$

Problem 4.3

Let $f(t)$ and $g(t)$ be given as follows:



(a) Sketch the function: $x(t) = f(t) * f(t)$

(b) Show that in general (hint: take the Fourier Transform of both sides):
if $a(t) = b(t) * c(t)$, then $b(t - t_0) * c(t) = a(t - t_0)$.

(c) Show that in general (hint: use the convolution integral formula):
if $a(t) = b(t) * c(t)$, then $(Mb(t)) * c(t) = Ma(t)$, for any real number M.

(d) Show that in general:
$a(t) * (b(t) + c(t)) = a(t) * b(t) + a(t) * c(t)$
(i.e. convolution is *distributive* with respect to addition)

(e) Write $g(t)$ in terms of $f(t)$ and use the three previous properties to solve $y(t) = f(t) * g(t)$ in terms of x(t) from part a.

(f) Solve and then sketch the function $z(t) = g(t + 2) * g(t)$ (hint: use shifted versions of x(t) from part a).

Solutions

(a) We have

$$x(t) = f(t) * f(t) = \int_{-\infty}^{\infty} f(\tau) f(t-\tau)\, d\tau = \begin{cases} \int_0^t 1\, d\tau & \text{if } 0 \le t < 1 \\ \int_{t-1}^1 1\, d\tau & \text{if } 1 \le t < 2 \\ 0 & \text{otherwise.} \end{cases}$$

$$= \begin{cases} t & \text{if } 0 \le t < 1 \\ 2-t & \text{if } 1 \le t < 2 \\ 0 & \text{otherwise.} \end{cases}$$

$$= 2\,\Delta\left(\frac{t-1}{2}\right)$$

(b) Assume $a(t) = b(t) * c(t)$. Let $t_0 \ge 0$. Then $A(\omega) = B(\omega)C(\omega)$, and

$$\mathcal{F}(b(t-t_0) * c(t)) = \mathcal{F}(b(t-t_0))\, \mathcal{F}(c(t)) = e^{-j\omega t_0}\, B(\omega)C(\omega) = e^{-j\omega t_0}\, A(\omega) = \mathcal{F}(a(t-t_0)).$$

Hence $b(t-t_0) * c(t) = a(t-t_0)$.

(c) Assume $a(t) = b(t) * c(t)$. Let M be a real number. Then

$$(Mb(t)) * c(t) = \int_{-\infty}^{\infty} (Mb(t-\tau))\, c(\tau)\, d\tau = M \int_{-\infty}^{\infty} b(t-\tau)\, c(\tau)\, d\tau = Ma(t).$$

(d) We have

$$\begin{aligned} a(t) * (b(t) + c(t)) &= \int_{-\infty}^{\infty} a(t-\tau)\, (b(\tau) + c(\tau))\, d\tau \\ &= \int_{-\infty}^{\infty} a(t-\tau)\, b(\tau)\, d\tau + \int_{-\infty}^{\infty} a(t-\tau)\, c(\tau)\, d\tau = a(t) * b(t) + a(t) * c(t). \end{aligned}$$

(e) We have $g(t) = f(t) - f(t-2) + f(t-4)$, so

$$f(t) * g(t) = f(t) * f(t) - f(t) * f(t-2) + f(t) * f(t-4) = x(t) - x(t-2) + x(t-4).$$

(f) We have $g(t) = f(t) - f(t-2) + f(t-4)$ and $g(t+2) = f(t+2) - f(t) + f(t-2)$, so

$$\begin{aligned} g(t) * g(t+2) &= f(t) * f(t+2) - f(t) * f(t) + f(t) * f(t-2) \\ &\quad - f(t-2) * f(t+2) + f(t-2) * f(t) - f(t-2) * f(t-2) \\ &\quad + f(t-4) * f(t+2) - f(t-4) * f(t) + f(t-4) * f(t-2) \\ &= x(t+2) - x(t) + x(t-2) \\ &\quad - x(t) + x(t-2) - x(t-4) \\ &\quad + x(t-2) - x(t-4) + x(t-6) \\ &= x(t-6) - 2x(t-4) + 3x(t-3) - 2x(t) + x(t+2) \end{aligned}$$

Problem 4.4

When the input to an LTI system is the *unit step function* $u(t)$, the output is $r(t) = \begin{cases} 0 & \text{if } t < 0 \\ t & \text{if } 0 \le t \le 1 \\ 1 & \text{if } t > 1. \end{cases}$

(a) Let $\epsilon > 0$. Sketch the output $y_\epsilon(t)$ when the input to the system is $x_\epsilon(t) = \begin{cases} 1/\epsilon & \text{if } 0 < t < \epsilon \\ 0 & \text{otherwise} \end{cases}$

Hint: Write $x_\epsilon(t)$ in terms of $u(t)$, and use the fact the system is linear and time-invariant.

(b) Evaluate $\lim\limits_{\epsilon \to 0} y_\epsilon(t)$.

(c) Evaluate $\lim\limits_{\epsilon \to 0} x_\epsilon(t)$. Why is the impulse response $h(t)$ equal to $\lim\limits_{\epsilon \to 0} y_\epsilon(t)$?

(d) Using the properties of the Fourier transform, prove:
if $z(t) = f(t) * g(t)$, then $\frac{df(t)}{dt} * g(t) = \frac{dz(t)}{dt}$.

(e) In general, if $s(t)$ is the output of an LTI system when $u(t)$ is the input, what is the impulse response $h(t)$? ($s(t)$ is also known as the *unit step response*)

Solution:

(a) We have

$$x_\epsilon(t) = \frac{1}{\epsilon}(u(t) - u(t-\epsilon)).$$

By the linearity and time-invariance of the system, we have

$$x_\epsilon(t) = \frac{1}{\epsilon}(u(t) - u(t-\epsilon)) \longrightarrow \text{ System } \longrightarrow \frac{1}{\epsilon}(r(t) - r(t-\epsilon)) = y_\epsilon(t).$$

Hence

$$\begin{aligned} y_\epsilon(t) &= \left(\begin{cases} 0 & \text{if } t < 0 \\ t/\epsilon & \text{if } 0 \le t \le 1 \\ 1/\epsilon & \text{if } t > 1 \end{cases} \right) - \left(\begin{cases} 0 & \text{if } t < \epsilon \\ (t-\epsilon)/\epsilon & \text{if } \epsilon \le t \le 1+\epsilon \\ 1/\epsilon & \text{if } t > 1+\epsilon \end{cases} \right) \\ &= \begin{cases} 0 & \text{if } t < 0 \\ t/\epsilon & \text{if } 0 \le t < \epsilon \\ 1 & \text{if } \epsilon \le t < 1 \\ (1+\epsilon-t)/\epsilon & \text{if } 1 \le t < 1+\epsilon \\ 0 & \text{if } t \ge 1. \end{cases} \end{aligned}$$

(b) We have

$$\lim_{\epsilon \to 0} y_\epsilon(t) = \begin{cases} 0 & \text{if } t < 0 \\ 1 & \text{if } 0 < t < 1 \\ 0 & \text{if } t > 1. \end{cases}$$

(c) We have

$$\delta(t) = \frac{d}{dt}u(t) = \lim_{\epsilon\to 0}\frac{u(t)-u(t-\epsilon)}{\epsilon} = \lim_{\epsilon\to 0} x_\epsilon(t)$$

so

$$\delta(t) = \lim_{\epsilon\to 0} x_\epsilon(t) \longrightarrow \boxed{\text{System}} \longrightarrow \lim_{\epsilon\to 0} y_\epsilon(t) = h(t).$$

(d) Suppose $z(t) = f(t) * g(t)$. Then we have $Z(\omega) = F(\omega)G(\omega)$. Now consider the Fourier transform of the convolution

$$\mathcal{F}\left(\frac{df(t)}{dt} * g(t)\right) = \mathcal{F}\left(\frac{df(t)}{dt}\right)\mathcal{F}\left(g(t)\right) = j\omega\, F(\omega)\, G(\omega) = j\omega\, Z(\omega) = \mathcal{F}\left(\frac{dz(t)}{dt}\right).$$

Hence when $z(t) = f(t) * g(t)$, we have $\frac{df(t)}{dt} * g(t) = \frac{dz(t)}{dt}$.

(e) Now since

$$s(t) = h(t) * u(t)$$

we have

$$h(t) = h(t) * \delta(t) = h(t) * \frac{du(t)}{dt} = \frac{ds(t)}{dt}.$$

Thus the impulse response is the derivative of the unit-step response.

Problem 4.5

Calculate $y(t) = x(t) * h(t)$ when

(a) $x(t) = e^{-t}u(t)$ and $h(t) = u(t)$

(b) $x(t) = e^{-t}u(t)$ and $h(t) = \begin{cases} 1 & \text{if } |t| < 1 \\ 0 & \text{otherwise.} \end{cases}$

Solutions

(a) We have

$$\begin{aligned} y_1(t) = e^{-t}u(t) * u(t) &= \int_{-\infty}^{\infty} e^{-\tau}u(\tau)u(t-\tau)\, d\tau \\ &= \int_{-\infty}^{t} e^{-\tau}\, u(\tau)\, d\tau = \begin{cases} \int_0^t e^{-\tau}\, d\tau & \text{if } t \geq 0 \\ 0 & \text{if } t < 0 \end{cases} \\ &= u(t)\int_0^t e^{-\tau}\, d\tau = u(t)\left(1 - e^{-t}\right) \end{aligned}$$

(b) We have $h(t) = u(t+1) - u(t-1)$ in this case, so by the distributive and time-shift properties of convolution

$$\begin{aligned} y_2(t) &= e^{-t}u(t) * u(t+1) - e^{-t}u(t) * u(t-1) = y_1(t+1) - y_1(t-1) \\ &= u(t+1)\left(1 - e^{-t-1}\right) - u(t-1)\left(1 - e^{-t+1}\right) = \begin{cases} 0 & \text{if } t < -1 \\ 1 - e^{-t-1} & \text{if } -1 \leq t < 1 \\ e^{-t}(e - 1/e) & \text{if } t \geq 1 \end{cases} \end{aligned}$$

Problem 4.6

The output of an LTI system is $y(t) = \begin{cases} 1 & \text{if } 3 \le t \le 5 \\ 0 & \text{otherwise.} \end{cases}$

Determine the impulse response $h(t)$ when the input $f(t)$ is

(a) $f(t) = \begin{cases} 1 & \text{if } 0 \le t \le 2 \\ 0 & \text{otherwise} \end{cases}$

(b) $f(t) = 2u(t)$

(c) $f(t) = \begin{cases} 1 & \text{if } 0 \le t \le 1 \\ 0 & \text{otherwise} \end{cases}$

Solutions

(a) We have

$$f(t) * h(t) = y(t) = f(t-3) = f(t) * \delta(t-3)$$

so $h(t) = \delta(t-3)$.

(b) We have

$$f(t) * h(t) = y(t) = u(t-3) - u(t-5) = \frac{f(t-3) - f(t-5)}{2} = f(t) * \frac{1}{2}(\delta(t-3) - \delta(t-5))$$

so $h(t) = \frac{1}{2}(\delta(t-3) - \delta(t-5))$.

(c) We have

$$f(t) * h(t) = y(t) = f(t-3) + f(t-4) = f(t) * (\delta(t-3) + \delta(t-4))$$

so $h(t) = \delta(t-3) + \delta(t-4)$.

Problem 4.7

Determine the *Nyquist rate* of the following signals:

(a) $a(t) = \dfrac{\sin(20t)}{t}$

(b) $b(t) = \cos^2(30t)\dfrac{\sin(20t)}{t}$

(c) $c(t) = \cos(10t)\dfrac{\sin(20t)}{t}$

(d) $d(t) = \left(\dfrac{\sin(20t)}{t}\right)^2$

(e) $e(t) = \left(\dfrac{\sin(20t)}{t}\right)^2 * \dfrac{\sin(2t)}{\pi t}$

Solutions

The Nyquist rate is twice the maximum frequency present in the signal. Specifically, let ω_m be the smallest frequency such that $X(\omega) = 0$ for all $|\omega| > \omega_m$. Then the Nyquist rate is $2\omega_m$.

(a)

$$a(t) = \frac{\sin(20t)}{t} \longleftrightarrow A(\omega) = \begin{cases} \pi & \text{if } |\omega| < 20 \\ 0 & \text{otherwise.} \end{cases}$$

We have $A(\omega) = 0$ for all $|\omega| > 20$, so the Nyquist rate is 40.

(b) We have

$$b(t) = \cos^2(30t)\frac{\sin(20t)}{t} = \frac{e^{-j60t}}{4}\frac{\sin(20t)}{t} + \frac{e^{j60t}}{4}\frac{\sin(20t)}{t} + \frac{1}{2}\frac{\sin(20t)}{t}$$

so by the frequency shifting property of the Fourier transform

$$B(\omega) = \left(\begin{cases} \pi/4 & \text{if } |\omega - 60| < 20 \\ 0 & \text{otherwise.} \end{cases}\right) + \left(\begin{cases} \pi/4 & \text{if } |\omega + 60| < 20 \\ 0 & \text{otherwise.} \end{cases}\right) + \left(\begin{cases} \pi/2 & \text{if } |\omega| < 20 \\ 0 & \text{otherwise.} \end{cases}\right)$$

$$= \begin{cases} \pi/4 & \text{if } 40 < |\omega| < 80 \\ \pi/2 & \text{if } |\omega| < 20 \\ 0 & \text{otherwise.} \end{cases}$$

So the Nyquist rate is 160.

(c) We have

$$c(t) = \cos(10t)\frac{\sin(20t)}{t} = \frac{e^{j10t}}{2}\frac{\sin(20t)}{t} + \frac{e^{-j10t}}{2}\frac{\sin(20t)}{t}$$

so by the frequency shifting property of the Fourier transform

$$C(\omega) = \left(\begin{cases} \pi/2 & \text{if } |\omega - 10| < 20 \\ 0 & \text{otherwise.} \end{cases}\right) + \left(\begin{cases} \pi/2 & \text{if } |\omega + 10| < 20 \\ 0 & \text{otherwise.} \end{cases}\right)$$

$$= \begin{cases} \pi & \text{if } |\omega| < 10 \\ \pi/2 & \text{if } 10 < |\omega| < 30 \\ 0 & \text{otherwise.} \end{cases}$$

So the Nyquist rate is 60.

(d) Define $z(\omega)$ as follows:

$$z(\omega) = u(\omega) * u(\omega) = \int_{-\infty}^{\infty} u(\omega - \theta)\, u(\theta)\, d\theta = \int_{-\infty}^{\omega} u(\theta)\, d\theta = \begin{cases} \omega & \text{if } \omega \geq 0 \\ 0 & \text{otherwise.} \end{cases} = \omega\, u(\omega).$$

Then

$$
\begin{aligned}
D(\omega) &= \frac{1}{2\pi}\left(\begin{cases}\pi & \text{if } |\omega| < 20\\ 0 & \text{otherwise.}\end{cases}\right) * \left(\begin{cases}\pi & \text{if } |\omega| < 20\\ 0 & \text{otherwise.}\end{cases}\right)\\
&= \frac{\pi}{2}\left(u(\omega-20)-u(\omega+20)\right) * \left(u(\omega-20)-u(\omega+20)\right)\\
&= \frac{\pi}{2}\left(u(\omega-20)*u(\omega-20)-2u(\omega-20)*u(\omega+20)+u(\omega+20)*u(\omega+20)\right)\\
&= \frac{\pi}{2}\left(Z(\omega-40)-2Z(\omega)+Z(\omega+40)\right)\\
&= \frac{\pi}{2}\begin{cases}40+\omega & \text{if } -40 \le \omega \le 0\\ 40-\omega & \text{if } 0 \le \omega < 40\\ 0 & \text{otherwise}\end{cases}\\
&= 20\pi\,\Delta\left(\frac{\omega}{80}\right)
\end{aligned}
$$

so the Nyquist rate is 80.

(e)

$$e(t) = d(t) * \frac{\sin(2t)}{\pi t}$$

so

$$
\begin{aligned}
E(\omega) &= D(\omega)\,\mathcal{F}\left(\frac{\sin(2t)}{\pi t}\right)\\
&= D(\omega)\begin{cases}1 & \text{if } |\omega| < 2\\ 0 & \text{otherwise}\end{cases}\\
&= \begin{cases}40+\omega & \text{if } -2 \le \omega \le 0\\ 40-\omega & \text{if } 0 \le \omega < 2\\ 0 & \text{otherwise}\end{cases}
\end{aligned}
$$

so the Nyquist rate is 4.

Problem 4.8

Let $r(t) = \displaystyle\sum_{k=-\infty}^{\infty} \delta(t-kT)$

(a) Show that $r(t)$ is periodic with period T.

(b) Calculate the Fourier series components R_n of $r(t)$ and write $r(t)$ as its exponential Fourier series.

(c) Show the Fourier transform $R(\omega)$ of $r(t)$ can be written as a sum of complex exponentials or as a sum of impulse functions.

Recall that $\mathcal{F}(e^{j\omega_0 t}) = 2\pi\delta(\omega-\omega_0)$.

(d) Let $x(t)$ be an arbitrary signal, and let $s(t) = x(t)r(t)$. Why is $s(t)$ a reasonable way to mathematically model a sampled signal?

(e) Write the Fourier transform $S(\omega)$ of $s(t)$ in terms of the Fourier transform $X(\omega)$ of $x(t)$.

Solutions

(a) We have

$$r(t-T) = \sum_{k=-\infty}^{\infty} \delta(t-T-kT) = \sum_{k=-\infty}^{\infty} \delta(t-(k+1)T) = \sum_{k=-\infty}^{\infty} \delta(t-kT) = r(t).$$

(b) In the period $[-T/2, T/2)$, $r(t) = \delta(t)$, so

$$R_n = \frac{1}{T}\int_T r(t)e^{-j\omega_0 nt}\,dt = \frac{1}{T}\int_{-T/2}^{T/2} \delta(t)e^{-j\omega_0 nt}\,dt = \frac{1}{T}\int_{-T/2}^{T/2} \delta(t)\,dt = \frac{1}{T}.$$

Hence, another way to write $r(t)$ is as its exponential Fourier series

$$r(t) = \frac{1}{T}\sum_{n=-\infty}^{\infty} e^{j\frac{2\pi}{T}nt}$$

(c) In general, if $f(t)$ is periodic with period T and Fourier series components F_n, then the Fourier transform of $f(t)$ is

$$F(\omega) = \mathcal{F}\left(\sum_{n=-\infty}^{\infty} F_n e^{j\omega_0 nt}\right) = \sum_{n=-\infty}^{\infty} F_n\,\mathcal{F}(e^{j\omega_0 nt}) = 2\pi\sum_{n=-\infty}^{\infty} F_n\,\delta(\omega-\omega_0 n).$$

Hence

$$R(\omega) = \frac{2\pi}{T}\sum_{n=-\infty}^{\infty} \delta\left(\omega - \frac{2\pi}{T}\right).$$

Alternatively, if we use the fact $\mathcal{F}(\delta(t-t_0)) = e^{-j\omega t_0}$, then

$$F(\omega) = \mathcal{F}\left(\sum_{k=-\infty}^{\infty} \delta(t-kT)\right) = \sum_{k=-\infty}^{\infty} \mathcal{F}\left(\delta(t-kT)\right) = \sum_{k=-\infty}^{\infty} e^{-j\omega kT}.$$

(d) We have

$$s(t) = x(t)r(t) = x(t)\sum_{k=-\infty}^{\infty}\delta(t-kT) = \sum_{k=-\infty}^{\infty} x(t)\delta(t-kT) = \sum_{k=-\infty}^{\infty} x(kT)\delta(t-kT)$$

so $s(t)$ "picks out" the values of $x(t)$ at integer multiples of the sampling period and is equal to 0 everywhere else. This is exactly what happens when we sample a signal. We pick out the values at certain times and "throw away" the rest of the signal.

(e) Since $s(t) = x(t)r(t)$, we have

$$S(\omega) = \frac{1}{2\pi}X(\omega) * R(\omega) = \frac{1}{T}\sum_{n=-\infty}^{\infty} X(\omega) * \delta\left(\omega - \frac{2\pi}{T}\right) = \frac{1}{T}\sum_{n=-\infty}^{\infty} X\left(\omega - \frac{2\pi}{T}\right)$$

Problem 4.9

Suppose a signal

$$x(t) = \frac{\sin(\pi t)}{\pi t}$$

is sampled with period T to form a new continuous-time signal $s(t)$ by taking

$$s(t) = \sum_{n=-\infty}^{\infty} x(nT)\, \delta(t - nT)$$

and the signal $y(t)$ is formed by taking

$$y(t) = T\, s(t) * \frac{\sin(\pi t)}{\pi t}.$$

Sketch $S(\omega) = \mathcal{F}(s(t))$ and determine $y(t)$ in the following cases:

(a) $T = 1/2$

(b) $T = 2$

(c) $T = 4/3$

What is the Nyquist rate for $x(t)$? How does this explain the resulting $y(t)$'s in (a), (b), and (c)?

Solutions

We have

$$X(\omega) = \begin{cases} 1 & \text{if } |\omega| < \pi \\ 0 & \text{otherwise.} \end{cases}$$

and

$$S(\omega) = \frac{1}{T} \sum_{k=-\infty}^{\infty} X(\omega + \frac{2\pi k}{T})$$

and

$$Y(\omega) = H(\omega) S(\omega) = \left(\begin{cases} 1 & \text{if } |\omega| < \pi \\ 0 & \text{otherwise.} \end{cases} \right) \sum_{k=-\infty}^{\infty} X(\omega + \frac{2\pi}{T})$$

(a) When $T = 1/2$, we have

$$X(\omega + 4\pi k) = \begin{cases} 1 & \text{if } \pi(4k-1) < \omega < \pi(4k+1) \\ 0 & \text{otherwise} \end{cases}$$

and the periods of $X(\omega + 4\pi k)$ do not interfere with each other, since

$$\pi(1 + 4k) < \pi(4(k + 1) - 1)$$

for all k.

Thus

$$Y(\omega) = X(\omega)$$

so $y(t) = \dfrac{\sin(\pi t)}{\pi t}$.

(b) When $T = 2$, we have

$$X(\omega + \pi k) = \begin{cases} 1 & \text{if } \pi(k-1) < \omega < \pi(k+1) \\ 0 & \text{otherwise} \end{cases}$$

so the periods of $X(\omega + \pi k)$ do interfere with each other. In particular, we have

$$S(\omega) = 1$$

for all ω, and

$$Y(\omega) = \begin{cases} 2 & \text{if } |\omega| < \pi \\ 0 & \text{otherwise.} \end{cases} = 2\,X(\omega)$$

which implies $y(t) = 2\,\dfrac{\sin(\pi t)}{\pi t}$.

(c) When $T = 3/4$, we have

$$X(\omega + 3\pi k/2) = \begin{cases} 1 & \text{if } \pi(3k/2 - 1) < \omega < \pi(3k/2 + 1) \\ 0 & \text{otherwise} \end{cases}$$

so the periods of $X(\omega + 3\pi k/2)$ do interfere with each other. In particular, we have (plot $S(\omega)$) so

$$Y(\omega) = \begin{cases} 1 & \text{if } |\omega| < \pi/2 \\ 2 & \text{if } \pi/2 < |\omega| < \pi \\ 0 & \text{otherwise.} \end{cases}$$

There are numerous ways of writing $y(t)$, but one such way is

$$y(t) = 2\,\frac{\sin(\pi t)}{\pi t} - \frac{\sin(\pi t/2)}{\pi t}$$

The maximum frequency in $x(t)$ is π, so the Nyquist rate is 2π. The sampling theorem implies that when we sample a rate above 2π, we can perfectly reconstruct $x(t)$ from its samples, but if the sampling rate is below 2π, there will be aliasing. In (a), we sample at rate $2\pi/T = 4\pi$, and we reconstruct the signal. In (b), we sample at rate $2\pi/T = \pi$, so there is aliasing. In (c), we sample at rate $2\pi/T = 3\pi/2$, so there is aliasing.