ECE 45 Homework 4

Problem 4.1

Simplify the following expressions as much as possible:

(a) $a(t) = (1 + t^2)\,(\delta(t) - 2\delta(t-2))$

(b) $b(t) = \cos(2\pi t)\,\left(\frac{du(t)}{dt} + \delta(t + 1/4)\right)$

(c) $c(t) = \sin(2\pi t)\,\delta(1/2 - 2t)$

(d) $d(t) = \int_{-6}^{\infty}(\tau^2 + 6)\,\delta(\tau - 2)\,d\tau$

(e) $e(t) = \int_{6}^{\infty}(\tau^2 + 6)\,\delta(\tau - 2)\,d\tau$

(f) $f(t) = \int_{-\infty}^{t}\delta(\tau - 2)\,d\tau$

(g) $g(t) = u(t) * (\delta(t+2) - \delta(t-2))$

Please report any typos/errors to j2connelly@ucsd.edu

(g) $g(t) = u(t) * \delta(t+2) - u(t) * \delta(t-2) = u(t+2) - u(t-2) = \begin{cases} 1 & \text{if } -2 \leq t < 2 \\ 0 & \text{otherwise.} \end{cases}$

Problem 4.2

Let $f(t) = \begin{cases} 2 - |t| & \text{if } |t| \leq 2 \\ 0 & \text{otherwise.} \end{cases}$

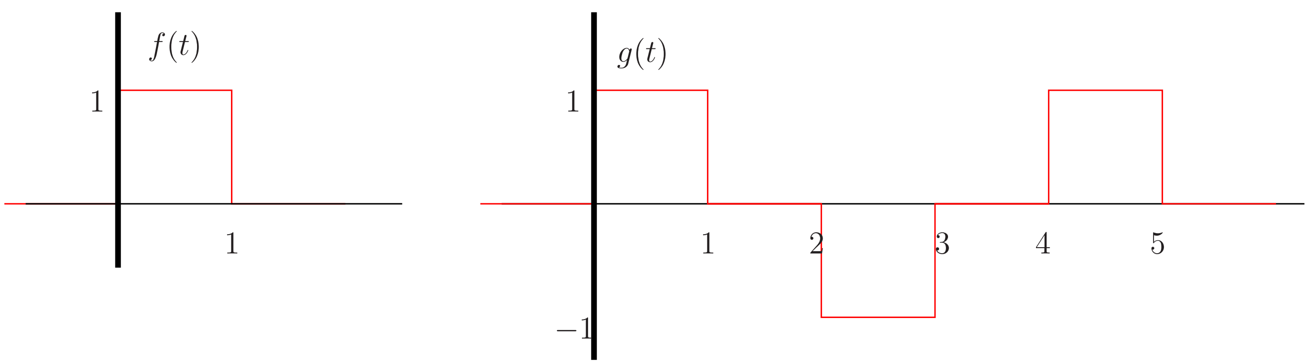
(a) Determine the function $y(t) = f(t) * u(t)$.

(b) Determine the function $z(t) = \frac{df(t)}{dt} * u(t)$.

(c) Determine the function $w(t) = f(t) * \frac{du(t)}{dt}$.

Problem 4.3

Let $f(t)$ and $g(t)$ be given as follows:



(a) Sketch the function: $x(t) = f(t) * f(t)$

(b) Show that in general (hint: take the Fourier Transform of both sides):
if $a(t) = b(t) * c(t)$, then $b(t - t_0) * c(t) = a(t - t_0)$.

(c) Show that in general (hint: use the convolution integral formula):
if $a(t) = b(t) * c(t)$, then $(Mb(t)) * c(t) = Ma(t)$, for any real number M.

(d) Show that in general:
$a(t) * (b(t) + c(t)) = a(t) * b(t) + a(t) * c(t)$
(i.e. convolution is *distributive* with respect to addition)

(e) Write $g(t)$ in terms of $f(t)$ and use the three previous properties to solve $y(t) = f(t) * g(t)$ in terms of x(t) from part a.

(f) Solve and then sketch the function $z(t) = g(t + 2) * g(t)$ (hint: use shifted versions of x(t) from part a).

Problem 4.4

When the input to an LTI system is the *unit step function* $u(t)$, the output is $r(t) = \begin{cases} 0 & \text{if } t < 0 \\ t & \text{if } 0 \le t \le 1 \\ 1 & \text{if } t > 1. \end{cases}$

(a) Let $\epsilon > 0$. Sketch the output $y_\epsilon(t)$ when the input to the system is $x_\epsilon(t) = \begin{cases} 1/\epsilon & \text{if } 0 < t < \epsilon \\ 0 & \text{otherwise} \end{cases}$

Hint: Write $x_\epsilon(t)$ in terms of $u(t)$, and use the fact the system is linear and time-invariant.

(b) Evaluate $\lim_{\epsilon \to 0} y_\epsilon(t)$.

(c) Evaluate $\lim_{\epsilon \to 0} x_\epsilon(t)$. Why is the impulse response $h(t)$ equal to $\lim_{\epsilon \to 0} y_\epsilon(t)$?

(d) Using the properties of the Fourier transform, prove:
if $z(t) = f(t) * g(t)$, then $\frac{df(t)}{dt} * g(t) = \frac{dz(t)}{dt}$.

(e) In general, if $s(t)$ is the output of an LTI system when $u(t)$ is the input, what is the impulse response $h(t)$? ($s(t)$ is also known as the *unit step response*)

Problem 4.5

Calculate $y(t) = x(t) * h(t)$ when

(a) $x(t) = e^{-t}u(t)$ and $h(t) = u(t)$

(b) $x(t) = e^{-t}u(t)$ and $h(t) = \begin{cases} 1 & \text{if } |t| < 1 \\ 0 & \text{otherwise.} \end{cases}$

Problem 4.6

The output of an LTI system is $y(t) = \begin{cases} 1 & \text{if } 3 \leq t \leq 5 \\ 0 & \text{otherwise.} \end{cases}$

Determine the impulse response $h(t)$ when the input $f(t)$ is

(a) $f(t) = \begin{cases} 1 & \text{if } 0 \leq t \leq 2 \\ 0 & \text{otherwise} \end{cases}$

(b) $f(t) = 2u(t)$

(c) $f(t) = \begin{cases} 1 & \text{if } 0 \leq t \leq 1 \\ 0 & \text{otherwise} \end{cases}$

Problem 4.7

Determine the *Nyquist rate* of the following signals:

(a) $a(t) = \dfrac{\sin(20t)}{t}$

(b) $b(t) = \cos^2(30t)\,\dfrac{\sin(20t)}{t}$

(c) $c(t) = \cos(10t)\,\dfrac{\sin(20t)}{t}$

(d) $d(t) = \left(\dfrac{\sin(20t)}{t}\right)^2$

(e) $e(t) = \left(\dfrac{\sin(20t)}{t}\right)^2 * \dfrac{\sin(2t)}{\pi t}$

Problem 4.8

Let $r(t) = \sum_{k=-\infty}^{\infty} \delta(t - kT)$

(a) Show that $r(t)$ is periodic with period T.

(b) Calculate the Fourier series components R_n of $r(t)$ and write $r(t)$ as its exponential Fourier series.

(c) Show the Fourier transform $R(\omega)$ of $r(t)$ can be written as a sum of complex exponentials or as a sum of impulse functions.

Recall that $\mathcal{F}(e^{j\omega_0 t}) = 2\pi\delta(\omega - \omega_0)$.

(d) Let $x(t)$ be an arbitrary signal, and let $s(t) = x(t)r(t)$. Why is $s(t)$ a reasonable way to mathematically model a sampled signal?

(e) Write the Fourier transform $S(\omega)$ of $s(t)$ in terms of the Fourier transform $X(\omega)$ of $x(t)$.

Problem 4.9

Suppose a signal

$$x(t) = \frac{\sin(\pi t)}{\pi t}$$

is sampled with period T to form a new continuous-time signal $s(t)$ by taking

$$s(t) = \sum_{n=-\infty}^{\infty} x(nT)\,\delta(t - nT)$$

and the signal $y(t)$ is formed by taking

$$y(t) = T\,s(t) * \frac{\sin(\pi t)}{\pi t}.$$

Sketch $S(\omega) = \mathcal{F}(s(t))$ and determine $y(t)$ in the following cases:

(a) $T = 1/2$

(b) $T = 2$

(c) $T = 4/3$

What is the Nyquist rate for $x(t)$? How does this explain the resulting $y(t)$'s in (a), (b), and (c)?