Multi-armed bandit algorithms for crowdsourcing systems with online estimation of workers’ ability

1 Supplementary material

This section presents the proof of Theorem 4.1. We rewrite the Lemma from the paper:

Lemma 4.1. If the workers are sorted in decreasing order of their efficiencies \( e_k = v_k / c_k \), where 
\[
e_1 \geq e_2 \geq \ldots \geq e_K,
\]
then the optimal workers’ selection strategy for LP-BKP is 
\[
x_k^* = M_k \quad \forall k = 1, 2, \ldots, s - 1
\]
\[
x_s^* = \frac{B - \sum_{k=1}^{s-1} c_k M_k}{c_s}
\]
\[
x_k^* = 0 \quad \forall k = s + 1, \ldots, K.
\]
where the splitting worker \( s \) is such that 
\[
\sum_{k=1}^{s-1} c_k M_k \leq B \quad \text{and} \quad \sum_{k=1}^{s} c_k M_k > B.
\]
The maximum aggregated value contribution is 
\[
v_{LP-BKP}^* = \sum_{k=1}^{s} x_k^* v_k.
\]

Let \( i(n) \) be the worker selected at the \( n^{th} \) iteration of B-KUBE; \( B_n \) is the residual budget before the \( n^{th} \) iteration of B-KUBE; \( m_{(k,n)} \) is the remaining number of tasks a worker \( k \) can perform at the \( n^{th} \) iteration of B-KUBE; \( \hat{x}(n) \) is an estimate of \( \lfloor x_{B_n}^* \rfloor \) by DGA using the estimated efficiencies \( \hat{e}_k = \hat{w}_k / c_k \), where \( x_{B_n}^* \) is the solution proposed by Lemma 4.1 for a given budget \( B_n \) and the set \( \{m_{(k,n)}\} \) of the remaining tasks that can be performed by the workers; \( j \not\in \hat{x}(n) \) implies that an additional selection of worker \( j \) is not proposed by selection strategy \( \hat{x}(n) \); \( \hat{s}(n) \) is the estimated splitting worker by DGA at the \( n^{th} \) iteration; \( N_k(N) \) denotes the number of times the worker \( k \) is selected more than the number of selections proposed by \( \lfloor x^* \rfloor \) when B-KUBE stops after \( N \) iterations.

The following two lemmas are the key components of the proof of Theorem 4.2.

Lemma 2 Let B-KUBE perform \( N \) iterations. For all \( 1 \leq n \leq N \), if a worker \( j \) is selected, then 
\[
P(i(n) = j|N) \leq P(j \in \hat{x}(n)|N) + \left( \frac{C_{\text{max}}}{C_{\text{min}}} \right)^2 \frac{1}{N - n + 1}.
\] (1)

Proof: We consider the \( n^{th} \) iteration and assume that the estimated efficiencies of the workers \( \hat{e}_k = \hat{w}_k / c_k \) are such that \( \hat{e}_1 \geq \hat{e}_2 \geq \ldots \geq \hat{e}_K \). For convenience, we drop the conditioning on \( N \) in the notation. Let \( M^*(B_n, \{m_{(k,n)}\}) = \{m_{k,n}^*\} \), where \( m_{k,n}^* \) is the number of selections of worker \( k \)
The following inequalities can be obtained by combining eq. (3) and (4), and using the fact that $i(n)$ is independent of $B_n$, and $\{m_{(k,n)}\}$, given $M^*(B_n, \{m_{(k,n)}\})$, we have

\[
P(i(n) = j|B_n, \{m_{(k,n)}\}) = \sum_{\{m_{(k,n)}\}} P(i(n) = j|M^*(B_n, \{m_{(k,n)}\})) \cdot P(M^*(B_n, \{m_{(k,n)}\})|B_n, \{m_{(k,n)}\}).
\] (2)

DGA proposes the selection of the first $\hat{s}(n)-1$ workers up to their maximum remaining capacity $m_{(k,n)}$ and selects the worker $\hat{s}(n)$ as many times as feasible. These selections are same as the ones suggested by $\hat{x}(n)$. Since the selection strategies $\hat{x}(n)$ and $M^*(B_n, \{m_{(k,n)}\})$ are same for the first $\hat{s}(n)$ workers, the remaining budget after the selections of the first $\hat{s}(n)$ workers is at most $c_{\hat{s}(n)}$, otherwise, the worker $\hat{s}(n)$ can be selected one more time. Thus, the number of workers’ selections suggested by $M^*(B_n, \{m_{(k,n)}\})$ in addition to $\hat{x}(n)$ can be bounded as:

\[
\sum_{i \notin \hat{x}(n)} m_{i,n}^* \leq \frac{c_{\hat{s}(n)}}{C_{\min}}.
\] (3)

Additionally, the total number of selections as proposed by DGA can be bounded as:

\[
\sum_{k=1}^{K} m_{k,n}^* \geq \frac{B_n}{C_{\max}}.
\] (4)

The following inequalities can be obtained by combining eq. (3) and (4), and using the fact that $c_{\hat{s}(n)} \leq C_{\max}$,

\[
\frac{\sum_{i \notin \hat{x}(n)} m_{i,n}^*}{\sum_{k=1}^{K} m_{k,n}^*} \leq \frac{c_{\hat{s}(n)}}{C_{\min}} \frac{C_{\max}}{B_n} \leq \left(\frac{C_{\max}}{C_{\min}}\right)^2 \frac{C_{\min}}{B_n}.
\] (5)

Additionally, before each iteration $n$ of B-KUBE, the remaining number of iterations are $N - n + 1$. Therefore, the residual budget $B_n$ is at least $C_{\min}(N - n + 1)$. Thus, we have

\[
\frac{C_{\min}}{B_n} \leq \frac{1}{N - n + 1}.
\] (6)

Now, the probability on the right-hand side of eq. (2) can be written as

\[
P\left(i(n) = j|M^*(B_n, \{m_{(k,n)}\}) = \{m_{(k,n)}^*\}\right)
\]

\[
\overset{(a)}{=} P\left(i(n) = j, j \in \hat{x}(n)|M^*(B_n, \{m_{(k,n)}\}) = \{m_{(k,n)}^*\}\right)
\]

\[
+ P\left(i(n) = j, j \notin \hat{x}(n)|M^*(B_n, \{m_{(k,n)}\}) = \{m_{(k,n)}^*\}\right)
\]

\[
\overset{(b)}{\leq} P\left(j \in \hat{x}(n)|M^*(B_n, \{m_{(k,n)}\})\right)
\]

\[
+ \frac{1}{C_{\min}} \frac{C_{\max}}{B_n} \sum_{k=1}^{K} m_{k,n}^*
\]

\[
\overset{(c)}{=\leq} P\left(j \in \hat{x}(n)|M^*(B_n, \{m_{(k,n)}\})\right) + \frac{1}{C_{\min}} \frac{C_{\max}}{B_n} \sum_{k=1}^{K} m_{k,n}^*
\]

\[
\overset{(d)}{\leq} P\left(j \in \hat{x}(n)|M^*(B_n, \{m_{(k,n)}\})\right) + \frac{1}{C_{\min}} \frac{C_{\max}}{B_n} \sum_{k=1}^{K} m_{k,n}^*
\]

where (a) follows from the fact that two events are mutually exclusive; (b) follows because B-KUBE chooses worker $j$ with probability $m_{j,n}^*/\sum_{k=1}^{K} m_{(k,n)}$ and $j \notin \hat{x}(n)$; (c) follows because the probability is bounded by 1; (d) follows by combining eq. (5) and (6). The lemma follows by combining eq. (2) and (7), and using Bayes rule.
Lemma 3 Let \([x^*]\) be the optimal workers’ selection strategy. If \(j \notin [x^*]\) and \(j \in \hat{x}(n)\), then there is at least one worker \(k' \in [x^*]\) whose estimated efficiency is less than the estimated efficiency of the worker \(j\) i.e. \(\hat{e}_{k'} \leq \hat{e}_j\) and the worker \(k'\) can perform additional tasks.

Proof: If \([x^*]\) is the optimal workers’ selection strategy at budget \(B\), then for any budget \(B' < B\) the optimal selection strategy \([x^*_{B'}]\) is a subset of the selections proposed by \([x^*]\). This can be seen from Lemma 4.1.

We can say that if a worker \(j \notin [x^*]\), then \(j \notin [x^*_{B_n}]\) for the residual budget \(B_n\) as \(B_n \leq B\). Now \(\hat{x}(n)\) is an estimate of \([x^*_{B_n}]\) by DGA, and \(j \in \hat{x}(n)\) according to the hypothesis in this lemma. Thus, there is at least one worker \(k' \in [x^*_{B_n}]\) whose estimated efficiency is less than the estimated efficiency of worker \(j\) by DGA i.e. \(\hat{e}_{k'} \leq \hat{e}_j\). Also, \(k' \in [x^*_{B_n}]\) implies that the worker \(k'\) can still perform tasks. As \([x^*_{B_n}]\) is a subset of \([x^*]\) and \(k' \in [x^*_{B_n}]\), therefore \(k' \in [x^*]\) and the worker \(k'\) can perform more tasks. 

Theorem 4.2 For a given budget \(B\), let B-KUBE perform \(N\) iterations. Assume that \([x^*]\) is the optimal selection strategy for the workers. Then, the expected number of times a worker \(k\) is selected more than the number of selections proposed by \([x^*]\) is

\[
E[N_k(N)|N] \leq \left( \min \left\{ \frac{8}{Q_{\min}^2 d_s^2} + \left( \frac{C_{\max}}{C_{\min}} \right)^2 \right\} \log N \right.
+ \frac{\pi^2}{3} + 1,
\]

where

\[
Q_{\min} = \min_{k \notin I \cup \{s\}} |e_k - e_s| = \min_{k \notin I \cup \{s\}} \left| v_k/c_k - v_s/c_s \right|
\]

\(I^*\) is the set of the top \(s - 1\) workers, arranged in decreasing order of their efficiencies \(e_k\), \(s\) is the splitting worker; \(d_s = |v_{s-1}/c_{s-1} - v_s/c_s|\), \(C_{\max} = \max_{k \in [K]} c_k\) and \(C_{\min} = \min_{k \in [K]} c_k\).

Proof: Without loss of generality, let us assume that the efficiencies \(e_k\) of the workers are such that \(e_1 \geq e_2 \geq \ldots \geq e_K\). The notation of conditioning \(N\) is dropped for convenience.

If \([x^*]\) is the optimal selection strategy, then by Lemma 4.1 the selection of the first \(s - 1\) workers is always optimal. Thus, if a worker \(j \notin [x^*]\) then the worker \(j \geq s\). According to \([x^*]\), the selection of these workers is always sub-optimal with the exception of the \(s^{th}\) worker. Therefore, \(j = s\) will be handled separately in the proof. Thus, \(N_j(N)\) for \(j \notin [x^*]\) can be written as

\[
N_j(N) \leq 1 + \min \left\{ \sum_{n=K+1}^{N} \{i(n) = j\}, M_j \right\} \leq 1 + \sum_{n=K+1}^{N} \{i(n) = j\}.
\]

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Taking expectations on both sides, we have
\[
E[N_j(N)] \leq 1 + \sum_{n=K+1}^{N} P(i(n) = j)
\]
\[
\leq 1 + \sum_{n=K+1}^{N} P\left( j = \hat{x}(n) \right)
\]
\[
+ \sum_{n=K+1}^{N} \left( \frac{C_{\max}}{C_{\min}} \right)^2 \frac{1}{N-n+1}
\]
\[
(b) \leq l + \sum_{n=K+1}^{N} P\left( j = \hat{x}(n), M_j(n) \geq l \right)
\]
\[
+ \sum_{n=K+1}^{N} \left( \frac{C_{\max}}{C_{\min}} \right)^2 \frac{1}{N-n+1},
\]
for \(1 \leq l \leq M_j\), where (a) follows from Lemma 2; (b) follows from the intersection of events \( j \in \hat{x}(n) \) and \( M_j(n) \geq l \) where \( M_j(n) \) is the number of times worker \( j \) is selected before \( n \)th iteration i.e. \( M_j(n) = M_j - m_{(j,n)} \).

Let \( b_{N,m_{(j,n)}} = \sqrt{2 \log N/M_j - m_{(j,n)}} \) and \( b_{N,m_j} = \sqrt{2 \log N/M_j - m_j} \). Now, consider the event \( A(n, j) = \{ j \in \hat{x}(n), M_j(n) \geq l \} \) on the right-hand side of eq. (11). By hypothesis, we have \( j \notin [x^*]\), however, \( j \in \hat{x}(n)\), thus by Lemma 3 \( \exists k' \in [x^*] \) such that \( \hat{e}_{k'} \leq \hat{e}_j \). Note that Lemma 3 also accounts for the sub-optimal selections of the splitting worker \( s \). It follows that the probability of the event \( A(n, j) = \{ j \in \hat{x}(n), N_j(n) \geq l \} \) can be simplified as:

\[
\sum_{n=K+1}^{N} P\left( A(n, j) \right)
\]
\[
\leq \sum_{n=K+1}^{N} P\left( \hat{e}_j \geq \hat{e}_{k'}; M_j(n) \geq l \right)
\]
\[
= \sum_{n=K+1}^{N} P\left( \hat{v}_{j,m_{(j,n)}} \frac{b_{n,m_{(j,n)}}}{c_j} + \hat{v}_{k',m_{(j,n)}} \frac{b_{n,m_{(j,n)}}}{c_{k'}} \geq \frac{\hat{v}_{k',m_{(j,n)}}}{c_{k'}} + \frac{b_{n,m_{(j,n)}}}{c_{k'}}; M_j(n) \geq l \right)
\]
\[
\leq \sum_{n=K+1}^{N} P\left( \max_{l \leq m_j \leq \min{m_{(n,M_j)}}} \hat{v}_{j,m_j} \frac{b_{n,m_j}}{c_j} + \min_{1 \leq i \leq n} \left\{ \hat{v}_{k',m_{(j,i)}} \frac{b_{n,m_{(j,i)}}}{c_{k'}} \right\} \right)
\]
\[
\leq \sum_{n=1}^{N} \sum_{m_j=1}^{n} P(F),
\]
where the event \( F \) is defined as follows:

\[
\hat{v}_{j,m_j} \frac{b_{n,m_j}}{c_j} + \hat{v}_{k',m_{(j,i)}} \frac{b_{n,m_{(j,i)}}}{c_{k'}} + \hat{v}_{k',m_{(j,i)}} \frac{b_{n,m_{(j,i)}}}{c_{k'}}.
\]

The event \( F \) occurs only if at least one of the events among C, D and E occurs where

\[
C: \hat{v}_{k',m_{(j,i)}} \frac{b_{n,m_{(j,i)}}}{c_{k'}} \leq \frac{\hat{v}_{k'}}{c_{k'}}
\]
\[
D: \frac{\hat{v}_j}{c_j} \leq \hat{v}_{j,m_j} \frac{b_{n,m_j}}{c_j}
\]
\[ E: \quad \frac{v_{k'}}{c_{k'}} \leq \frac{v_j}{c_j} + 2 \frac{b_{n,m_j}}{c_j}. \] (15)

This claim can be proved by contradiction. Thus, the probability of the event \( F \) can be bounded as
\[ P(F) \leq P(C) + P(D) + P(E). \] (16)

Using Chernoff-Hoeffding inequalities, the probability of the events \( C \) and \( D \) can be bounded as
\[ P(C) \leq \exp\left( -2b^2_{(n,m_{k'},i')} \left( M_{k'} - m_{(k',i)} \right) \right) = n^{-4} \] (17)
\[ P(D) \leq \exp\left( -2b^2_{(n,m_j)} \left( M_j - m_j \right) \right) = n^{-4}. \] (18)

Next, we show that for \( l \geq 8 \log N/\min \{Q_{\min}^2, d_s^2\} \) the \( P(E) = 0 \). The analysis is split into two cases: \( j > s \) and \( j = s \).

**Case 1:** For \( j > s \) and \( l \geq 8 \log N/Q_{\min}^2 \), we have
\[ \frac{v_{k'}}{c_{k'}} - \frac{v_j}{c_j} - 2 \frac{b_{n,m_j}}{c_j} \geq \frac{v_{k'}}{c_{k'}} - \frac{v_j}{c_j} - 2 \sqrt{\frac{2 \log n}{l}}, \] (a)
\[ \frac{v_{k'}}{c_{k'}} - \frac{v_j}{c_j} - 2 \sqrt{\frac{2 \log n}{l}} \geq \frac{v_{k'}}{c_{k'}} - \frac{v_j}{c_j} - Q_{\min}, \] (b)
\[ \frac{v_{k'}}{c_{k'}} - \frac{v_j}{c_j} - Q_{\min} \geq \frac{v_{k'}}{c_{k'}} - \frac{v_j}{c_j} - \sqrt{\frac{2Q_{\min}^2 \log n}{8 \log N}}, \] (c)
\[ \frac{v_{k'}}{c_{k'}} - \frac{v_j}{c_j} - \sqrt{\frac{2Q_{\min}^2 \log n}{8 \log N}} \geq 0, \] (d)
\[ \frac{v_{k'}}{c_{k'}} - \frac{v_j}{c_j} - \sqrt{\frac{2Q_{\min}^2 \log n}{8 \log N}} \geq \frac{v_{k'}}{c_{k'}} - \frac{v_j}{c_j} - Q_{\min} + \sqrt{\frac{2Q_{\min}^2 \log n}{8 \log N}}, \] (e)
\[ \frac{v_{k'}}{c_{k'}} - \frac{v_j}{c_j} - Q_{\min} + \sqrt{\frac{2Q_{\min}^2 \log n}{8 \log N}} \geq 0, \] (f)

where (a) follows from the fact that \( \forall j \in [K], c_j \geq 1 \); (b) and (c) use the fact that \( m_j \geq l \geq 8 \log N/Q_{\min}^2 \); (d) follows from the fact that \( n \leq N \); (e) and (f) follow from the fact that \( Q_{\min} = \min_{j \in [K]} Q_j \) where \( Q_j = |v_s/c_s - v_j/c_j| \) and \( s \) is the splitting worker.

**Case 2:** For \( j = s \), we have by Lemma 3 that \( k' < s \). By following the same steps as in case 1, it can be shown that \( P(E) = 0 \) for \( l \geq 8 \log N/d_s^2 \) where \( d_s = |v_{s-1}/c_{s-1} - v_s/c_s| \).

Thus, for \( l \geq 8 \log N/\min \{Q_{\min}^2, d_s^2\} \), we have \( P(E) = 0 \). Now combining this fact with eq. (12), (17) and (18), we have
\[ \sum_{n=K+1}^N P\left( j \in \hat{x}(n), N_j(n) \geq l \right) \leq \sum_{n=1}^N \sum_{i=1}^n \sum_{m_j=1}^n 2n^{-4} \leq \frac{\pi^2}{3}. \] (20)

The third term in eq. (11) can be bounded as
\[ \sum_{n=K+1}^N \left( \frac{C_{\max}}{C_{\min}} \right)^2 \frac{1}{N - n + 1} \leq \left( \frac{C_{\max}}{C_{\min}} \right)^2 \log (N). \] (21)

Thus, for \( l = 8 \log N/\min \{Q_{\min}^2, d_s^2\} + 1 \), and combining eq. (11), (20) and (21) the result follows
\[ E[N_j(N)|N] \leq \min \left( \frac{8 \log N}{Q_{\min}^2, d_s^2} \right) + \left( \frac{C_{\max}}{C_{\min}} \right)^2 \log (N) \]
\[ + \frac{\pi^2}{3} + 1. \] (22)