CENG101A: Introductory Fluid Mechanics Fall Quarter 2006
http://maecourses.ucsd.edu/mae210a

Quiz II Solution

Daniel Bernoulli (1700–1782)

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1. Streamlines are everywhere parallel to the velocity field.
2. $\nabla \cdot \mathbf{x} = 3$ (in three dimensions).
3. The first law of thermodynamics governs the energy of a system.

2 Compute $\nabla \cdot \mathbf{u} = 0$ and $\nabla \times \mathbf{u} = (0, 0, 4)$. The flow is two-dimensional so the streamlines are in functions of x and y alone. The streamlines satisfy the equation $\mathrm{d}x/(-y) = \mathrm{d}y/(3x)$, which can be integrated to give $3x^2 + y^2 = C$, where C is a constant. Hence the streamline that passes through $(0, 1, 0)$ is $3x^2 + y^2 = 1$. This means that the curve $\psi =$ constant is a streamline so ψ does not change along the streamline. Finally $\nabla^2 \psi = 8$, which is twice the z-component of $\nabla \times \mathbf{u}$.

3 Assumptions: no viscous effects; constant density within each fluid layer; the tank is large enough that we may ignore the downward motion of the interfaces (by conservation of mass this velocity will be very small). Applying Bernoulli between the free surface (A), the interface (B), the end of the tube (C) and the top of the jet (D) gives (using gauge pressure)

$$gd_1 = \frac{p_B}{\rho_1} \quad \text{and} \quad \frac{p_B}{\rho_2} = \tfrac{1}{2} V_C^2 = gh.$$

(a) $V_C = \sqrt{2p_B/\rho_2} = \sqrt{2gd_1\rho_1/\rho_2}$. Plugging in numbers ($g = 9.81\ \mathrm{m\,s^{-2}}$) gives 4.43 $\mathrm{m\,s^{-1}}$.
(b) $h = \frac{1}{2}V_C^2/g = d_1\gamma_1/\gamma_2$. In numbers, 1 m.
(c) The momentum flux is $\rho_2 A V_C^2 = 2d_1 A\gamma_1$. In numbers 160 $\mathrm{kg\,m\,s^{-2}}$.

4 Assumptions: steady, no friction, constant density, hydrostatic pressure. Then mass conservation (per unit width) gives

$$\rho h_1 V_1 = \rho h_2 V_2.$$

The hydrostatic force (per unit width) at section 1 is $\frac{1}{2}\rho g h_1^2$ so Newton II gives

$$\tfrac{1}{2}\rho g h_1^2 - \tfrac{1}{2}\rho g h_2^2 = -\rho h_1 V_1^2 + \rho h_2 V_2^2.$$

(Note that there was a typo in the solution to Homework 3: it should have been h^2 on the left-hand side as can be seen from dimensional considerations.) Eliminate V_2 between the two equations to give

$$\tfrac{1}{2} g(h_1 - h_2)(h_1 + h_2) = -\frac{V_1^2 h_1}{h_2}(h_1 - h_2).$$

This is a cubic equation in h_2 but we can eliminate h_1 and obtain the quadratic

$$h_2^2 + h_1 h_2 + \frac{2V_1^2 h_1}{g} = 0.$$

This has a positive and a negative root: the positive one is physical and we obtain

$$h_2 = \frac{h_1}{2}\left[\left(1 + \frac{8V_1^2}{g h_1}\right)^{1/2} - 1\right].$$

Use the First Law with no heat change or work. Then since the system is steady $\int \rho(e + p/\rho)(\mathbf{u}\cdot\mathbf{n})\,\mathrm{d}A$ doesn't change. At section 1 this is

$$\int_0^{h_1} \rho\left[\frac{V_1^2}{2} + gz + u_1 + \frac{g(h_1 - z)}{2}\right]\mathrm{d}z = \rho\left(\frac{h_1 V_1^2}{2} + g h_1^2\right) + \int_0^{h_1} u_1\,\mathrm{d}z.$$

The gain in internal energy is

$$\int_0^{h_2} u_2\,\mathrm{d}z - \int_0^{h_1} u_1\,\mathrm{d}z = \rho\frac{h_1 V_1^2 - h_2 V_2^2}{2} + g\rho(h_1^2 - h_2^2) = \tfrac{1}{2} g\rho(h_1^2 - h_2^2)$$

which is non-zero since h_1 and h_2 are different.