CENG101A: Introductory Fluid Mechanics Fall Quarter 2006
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Quiz III Solution

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1. Newtonian fluids have a linear shear stress-rate of shear strain relation.
2. Fully-developed flow does not depend on the coordinate along the pipe (or channel).
3. The Navier–Stokes equation is derived from Newton's Second Law.

2 The vorticity of a two-dimensional flow is the z-direction, with $\omega = 2 - 2y$. Shear stress only has the $\tau_{xy} = \tau_{yx} = \mu(2 - 2y)$ components. The flow is incompressible and the Navier–Stokes equation is

$$\rho \frac{\mathrm{D}\mathbf{u}}{\mathrm{D}t} = \nabla p + \rho \mathbf{g} + \mu \nabla^2 \mathbf{u}.$$

The x- and y-components here become

$$0 = -\frac{\partial p}{\partial x} - 2\mu, \qquad 0 = -\frac{\partial p}{\partial y} - \rho g.$$

This question is wrong: there is an inconsistency between the pressure gradient and the velocity.

Either the pressure is wrong and should be $p = -2\mu x - y$, so that $\rho g = 1$. Then the pressure gradient is $(-2\mu, -1)$ and the velocities of the boundaries are $u(0) = 0$ and $u(1) = 1$. Or the velocity is wrong and should be Ay. Then the pressure gradient is $(0, 1)$ and the velocities of the boundaries are $u(0) = 0$ and $u(1) = A$. In both cases gravity acts downward.

3 As in class, we have

$$u = \frac{y^2}{2\mu}\frac{\mathrm{d}P}{\mathrm{d}x} + \left(Uh - \frac{h^2}{2\mu}\frac{\mathrm{d}P}{\mathrm{d}x}\right)y.$$

The average velocity is proportional to the volume flux

$$Q = \int_0^y u(y)\,\mathrm{d}y = U\frac{h^2}{2} - \frac{h^2}{12\mu}\frac{\mathrm{d}P}{\mathrm{d}x}.$$

For this to vanish, we need $U = (h^2/6\mu)\mathrm{d}P/\mathrm{d}x$.

4 Fully-developed, incompressible, steady flow of a Newtonian fluid in a pipe. A force balance on an annular control volume gives

$$r\frac{\mathrm{d}P}{\mathrm{d}x} = \frac{\mathrm{d}}{\mathrm{d}r}(r\tau_{rx}).$$

Integrate to give $\tau_{rx} = (r/2)\mathrm{d}P/\mathrm{d}x$, since we can't have a singularity at the axis. The pressure gradient is constant.

Now substitute in the constitutive relation for a Newtonian fluid:

$$\mu\frac{\mathrm{d}v_x}{\mathrm{d}r} = \frac{r}{2}\frac{\mathrm{d}P}{\mathrm{d}x}.$$

Integrate twice and use the no-slip boundary condition $v_x = 0$ at $r = a$:

$$v_x = \frac{1}{4\mu}\frac{\mathrm{d}P}{\mathrm{d}x}(r^2 - a^2).$$

The volume flux is then

$$Q = \int_0^a v_x 2\pi r\,\mathrm{d}r = \frac{\pi a^4}{8\mu}\frac{\mathrm{d}P}{\mathrm{d}x}.$$

The Hagen–Poiseuille law is this relationship, reexpressed in terms of the diameter D of the pipe and the pressure drop ΔP per length L:

$$\Delta P = \frac{128\mu Q}{\pi D^4}.$$