Quiz IV Solution

Edgar Buckingham (1867–1940)[1]

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1. The velocity potential exists only for irrotational flows.
2. The streamfunction exists only for incompressible flows.
3. The Buckingham Pi theorem generates dimensionless parameters.

2 The streamlines are parabolas about the y-axis. The vorticity is $\omega = -\nabla^2\psi = -2$. This does not vanish so the flow is not irrotational and there is no velocity potential.

3 Five parameters: Δp [ML^{-1}T^{-2}], D [L], ρ [ML^{-3}], ω [T^{-1}], Q [MT^{-1}]. Three dimensions. Two non-dimensional parameters. No unique choice; take ρ, D and ω as repeating parameters. Get

$$\pi_1 = \frac{\Delta p}{\rho D^2 \omega^2}, \qquad \pi_2 = \frac{Q}{\rho D^3 \omega}.$$

4 Two-dimensional incompressible irrotational flow: there exists a streamfunction Ψ with $\nabla^2\Psi = 0$. The boundary conditions are $\Psi \to Uy$ as $r \to \infty$ and $\Psi = C$ on $r = a$ (C is a constant). Try U times the solution given: it satisfies Laplace's equation. For large r, $\Psi \to Ur\sin\theta = Uy$. On the boundary of the cylinder, $\Psi = 0$. Hence it is the required solution.

The velocity on the boundary is purely azimuthal, with $u_\theta = \partial\Psi/\partial r_{r=a} = U\sin\theta(1 + a^2/r^2)_{r=a} = 2U\sin\theta$. The pressure field from Bernoulli is $p = p_\infty - 2\rho U^2 \sin^2\theta$. The drag is the force parallel to the flow, so we need to use $n_x = \cos\theta$. Then we have

$$D = -\int p n_x \, \mathrm{d}S = -\int_0^{2\pi} [p_\infty - 2\rho U^2 \sin^2\theta \cos\theta] a \, \mathrm{d}\theta = 0,$$

an example of d'Alembert's paradox.

[1]Actually the picture is Buckingham palace; I couldn't find a picture of Edgar Buckingham on the Internet.