Quiz V Solution

Ludwig Prandtl (1875–1953)

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1. In a boundary layer, there is no pressure gradient normal to the boundary.
2. The Reynolds stress is the turbulent contribution to shear stress.
3. The Fanning friction factor is a non-dimensional version of head loss.

2 In this experiment water flows from a large reservoir through a clear tube. A thin filament of dye is injected at the entrance to the tube allows visual observation of the flow. At low flow rates (low Reynolds number) the injected into the flow remains in a single filament; there is little dispersion of dye because the flow is laminar. A laminar flow is the one in which the fluid flows in laminae, or layers; there is no macroscopic mixing of adjacent fluid layers.

As the flow rate through the tube is increased, the dye filament becomes unstable and breaks up into a random motion; the line of dye is stretched and twisted into myriad entangled threads, and it quickly disperses throughout the entire flow field. This behavior of turbulent flow is due to small, high-frequency velocity fluctuations superimposed on the mean motion of a turbulent flow; the mixing of fluid particles from adjacent layers of fluid results in rapid dispersion of the dye.

The pipe flow regime is determined by the Reynolds number, Re $= \rho V D/\mu$. Under normal conditions, the transition from laminar to turbulent regime occurs at Re ≈ 2300.

From Fox & McDonald, "Introduction to Fluid Mechanics", 5th ed. Wiley. (p. 332-333).

3 The energy (extended Bernoulli) equation reduces to

$$w = g(y_1 - y_2) - h_L g$$

where w is the power (per mass flow rate). The Reynolds number is

$$\mathrm{Re} = \frac{\frac{1}{2}\times 4}{1.22\times 10^{-5}} = 164\,000.$$

Cast iron pipe so $e/D = 0.0017$. Then some playing around with the transition formula gives $f = 0.0059$. Hence $h_L = 1.401$ ft. The power is then

$$\dot{W} = \dot{m}w = \frac{g(-1-1.401)\mathrm{ft}}{550\mathrm{ft\ lb_f\ s^{-1}\ hp^{-1}}}\frac{62.3\mathrm{lb_m\ ft^{-3}}}{g\mathrm{lb_m lb_f^{-1}}}\times\frac{\pi}{4}\times(\tfrac{1}{2}\mathrm{ft})^2\times 4\mathrm{ft\ s^{-1}} = -0.2136\,\mathrm{hp}.$$

4 Find constants a and b. Boundary conditions: (i) $u = 0$ at $y = 0$: automatic; (ii) $u = U$ at $y = \delta$: $a\sin\delta = U$; (iii) $\mathrm{d}u/\mathrm{d}y = 0$ at $y = \delta$: $ab\cos b\delta = 0$; (iv) (optional) $\mathrm{d}^2u/\mathrm{d}y^2 = 0$ at $y = 0$: automatic. Condition (iii) gives $b\delta = n\pi/2$. The profile with $n = 1$ is the correct one (the others have reversed flow). Then (i) gives $a = U$. This gives $u = U\sin(\pi y/2\delta)$.

Now solve

$$\tau = \mu\left.\frac{\partial u}{\partial y}\right|_{y=0} = \rho U^2\frac{\mathrm{d}}{\mathrm{d}x}\int_0^\delta \frac{u}{U}\left(1-\frac{u}{U}\right)\,\mathrm{d}y.$$

This gives

$$\frac{\pi\mu U}{2\delta} = \int_0^\delta \sin\frac{\pi y}{2\delta}\left(1-\sin\frac{\pi y}{2\delta}\right)\,\mathrm{d}y = \frac{\mathrm{d}\delta}{\mathrm{d}x}\frac{4-\pi}{2\pi}.$$

Separate variables and get

$$\delta\mathrm{d}\delta = 11.5\frac{\mu}{\rho U}\mathrm{d}x;$$

Taking $\delta = 0$ at $x = 0$ gives

$$\frac{\delta}{x} = \frac{4.8}{\sqrt{\mathrm{Re}_x}}.$$

Momentum thickness:

$$\Theta = \int_0^\delta \frac{u}{U}\left(1-\frac{u}{U}\right)\,\mathrm{d}y = \int_0^\delta \sin\frac{\pi y}{2\delta}\left(1-\sin\frac{\pi y}{2\delta}\right)\,\mathrm{d}y = \delta\frac{4-\pi}{2\pi}.$$

Displacement thickness:

$$\delta^* = \int_0^\delta\left(1-\frac{u}{U}\right)\,\mathrm{d}y = \int_0^\delta\left(1-\sin\frac{\pi y}{2\delta}\right)\,\mathrm{d}y = \delta\frac{\pi-2}{\pi}.$$