Solutions I.

1 To test whether a substance is a fluid, see if it deforms continuously under a shear stress. Toothpaste can support a shear stress before deforming (this is known as a yield stress): you have to squeeze the tube hard enough before the toothpaste comes out. So toothpaste is not (just) a fluid. Shampoo on the other hand has no yield stress and is a fluid, although it has a more complicated rheology then fluids like water, oil and so on.

Glass: not a fluid at room temperature according to me. From a handout to MAE210A in F2005:

This has been the subject of much discussion on the Internet. From `http://math.ucr.edu/home/baez` "There is no clear answer to the question "Is glass solid or liquid?". In terms of molecular dynamics and thermodynamics it is possible to justify various different views that it is a highly viscous liquid, an amorphous solid, or simply that glass is another state of matter which is neither liquid nor solid. The difference is semantic. In terms of its material properties we can do little better. There is no clear definition of the distinction between solids and highly viscous liquids. All such phases or states of matter are idealisations of real material properties. Nevertheless, from a more common sense point of view, glass should be considered a solid since it is rigid according to every day experience. The use of the term "supercooled liquid" to describe glass still persists, but is considered by many to be an unfortunate misnomer that should be avoided. In any case, claims that glass panes in old windows have deformed due to glass flow have never been substantiated. Examples of Roman glassware and calculations based on measurements of glass visco-properties indicate that these claims cannot be true. The observed features are more easily explained as a result of the imperfect methods used to make glass window panes before the float glass process was invented."

See also `http://dwb.unl.edu/Teacher/NSF/C01/C01Links/www.ualberta.ca/~bderksen/florin.html` and E. D. Zanotto, Do cathedral glasses flow?, *Am. J. Phys.*, **66**, 392–395: "A general belief among members of the scientific community is that glass articles can be bent irreversibly and that they flow at ambient temperature. This myth is mostly based on widespread stories that stained-glass windows of medieval cathedrals are thicker in the lower parts. In this paper I estimate the time periods required for glass to flow and deform at ordinary temperatures, using calculated viscosity curves for several modern and ancient glass compositions. The conclusion is that window glasses may flow at ambient temperature only over incredibly long times, which exceed the limits of human history."

2 The Laplacian in plane polar coordinates is

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\phi}{\partial r}\right)+\frac{1}{r^2}\frac{\partial\partial^2\phi}{\partial\partial r^2}=0.$$

A radially symmetric field depends only on r, so we need to solve

$$\frac{\mathrm{d}}{\mathrm{d}r}\left(r\frac{\mathrm{d}\phi}{\mathrm{d}r}\right)=0.$$

This gives

$$r\frac{\mathrm{d}\phi}{\mathrm{d}r}=A \qquad \text{and hence} \qquad \phi=A\log r+B.$$

So the fields $\phi=$ constant and $\phi=\log r$ have vanishing Laplacian.

The Laplacian in spherical polar coordinates is messy, but the radial part is simply

$$\frac{1}{r^2}\frac{\mathrm{d}}{\mathrm{d}r}\left(r^2\frac{\mathrm{d}\phi}{\mathrm{d}r}\right).$$

Integrating this twice gives

$$r^2\frac{\mathrm{d}\phi}{\mathrm{d}r}=A \qquad \text{and hence} \qquad \phi=-\frac{A}{r}+B.$$

So the fields $\phi=$ constant and $\phi=r^{-1}$ have vanishing Laplacian.

3 The units of E are W m^{-2}, so E is power per unit area. Use 1 J = 0.000947817 Btu; 1 hr = 3600 s, 1 ft = 0.3048 m, 1 K = 1.8°R:

$$\sigma=5.676\times10^{-8}\times\frac{0.000947817\times3600}{0.3048^2\times1.8^4}=1.71\times10^{-9}\frac{\text{Btu}}{\text{hr ft}^2\ {}^\circ\text{R}^4}.$$

4 The atmosphere is at rest, so we solve

$$\frac{\mathrm{d}p}{\mathrm{d}z}=-\rho g.$$

Uniform density: integrate to get $p_E=p_a-\rho g z$. We need the density at sea level. From the ideal gas law

$$\rho=\frac{pm}{RT}=\frac{100\times10^3\times29\times10^{-3}}{8.3144\times(273.15+20)}\frac{\text{Pa kg}}{\text{J}}=1.19\ \text{kg m}^{-3}.$$

Hence $p_E=100\times10^3-1.19\times9.8\times8848=-3.17$ kPa. This is negative, which is obviously impossible.

Uniform temperature: combine the hydrostatic and ideal gas laws to give

$$\frac{\mathrm{d}p}{\mathrm{d}z}=-\frac{mg}{R}\frac{p}{T}.$$

If T is constant, this can be integrated to give

$$\log \frac{p}{p_a} = -\frac{mgz}{RT}.$$

Hence $p_E = 100 \times 10^3 \exp\left[-29 \times 10^{-3} \times 9.8 \times 8848/8.3144 \times (273.15 + 20)\right] = 35.6$ kPa. This is positive, which is better.

5 We don't know the shape of the vessel, so use the divergence theorem to get a volume integral:

$$\begin{aligned} F &= -\rho \int_S gz \, \mathrm{d}S_z = -\rho \int_S (0, 0, gz)\cdot \mathrm{d}\mathbf{S} = -\rho \int_V \nabla\cdot(0, 0, gz) \, \mathrm{d}V \\ &= -\rho \int_V g \, \mathrm{d}V = -\rho g V. \end{aligned}$$

The result is a scalar with units of force. The displacement of the vessel is in mass units, i.e. it is ρg. Use the US ton = 2000 pounds; then in SI units $F = 9.8 \times 7800 \times 2000 \times 0.454 = 69.4$ MN.