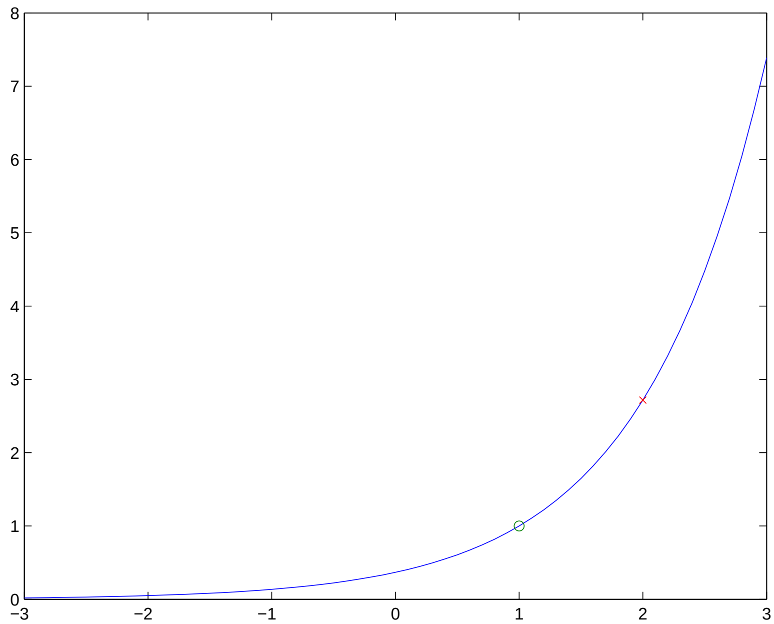
Solutions II.

1 The equation for the streamlines is $\mathrm{d}y/\mathrm{d}x = y/x$ which can be integrated to give $y = C\mathrm{e}^x$. The desired streamline passes through $(1, 1)$ so $1 = C\mathrm{e}$. Hence $y = \mathrm{e}^{x-1}$. The flow is steady so particle paths and streamlines are the same. We find that the point $(2, \mathrm{e})$ is on the same streamline with $x \geq 2$.



2 Compute the average velocity:

$$\bar{u} = \frac{1}{h}\int_0^h U\left(\frac{y}{h}\right)^n\left(1-\frac{y}{h}\right)\,\mathrm{d}y = \frac{U}{h^{n+1}}\left[\frac{y^{n+1}}{n+1} - \frac{y^{n+2}}{h(n+2)}\right]_0^h = \frac{U}{(n+1)(n+2)}.$$

To obtain $\bar{u} = U/3$, we need to solve the quadratic $n^2 + 3n + 2 = 3$. This has solutions $n = \frac{1}{2}(-3 \pm \sqrt{13})$, i.e. -3.3 and 0.2. The first one is physically impossible since the velocity would blow up at $y = 0$, so we have $n = \frac{1}{2}(-3 + \sqrt{13})$.

3 This is a steady problem with constant density, so we can work with volume fluxes. We find $Q = 56$ ft^3 s^{-1} entering, so the average speed out is Q divided by the total area of the windows, 12 ft^2. We obtain $v = 4.67$ ft s^{-1}. If the wind is at $45°$ to the door, the volume flux in is reduced to $56 \cos 45°$ ft^2 and the average speed out is reduced by the same amount, so $v = 3.30$ ft s^{-1}.

4 Write down conservation of mass of alcohol m_a in the form

$$\frac{\mathrm{d}m_a}{\mathrm{d}t} = 0.4q\rho_a - q\frac{m_a}{V}.$$

In this expression the first term is the mass flux of alcohol into the volume, since the volume of alcohol entering is 40% of the flow rate; the second term is the mass flux out. We solve this first-order linear ordinary differential equations with constant coefficients and find $m_a(t) = 0.4\rho_a V(1 - \mathrm{e}^{-qt/V})$, where we have used the result that $m_a(0) = 0$. We are asked to find the concentration of alcohol $C_a = m_a/(\rho_a V)$, which is given by $C_a = 0.4(1 - \mathrm{e}^{-qt/V})$. We are asked for t such that $C_a = 0.1$, so we solve

$$0.1 = 0.4(1 - \mathrm{e}^{-0.2t/10}),$$

so that $-0.02t = \ln 3/4$. Hence $t = 14.38$ minutes.

5 Conservation of mass:

$$0 = \frac{\partial}{\partial t}\int_V \rho \,\mathrm{d}V + \int_S \rho(\mathbf{v}\cdot\mathbf{n})\,\mathrm{d}S.$$

Assuming the velocity v and density ρ_1 are constant over the orifice gives

$$0 = \frac{\partial}{\partial t}\left(\frac{4\pi R^3}{3}\rho\right) + \pi a^2(-\rho_1 v).$$

Everything in the above expression is constant except ρ so we integrate to get

$$\rho = \rho_0 + \frac{3a^2 v\rho_1}{4R^3}$$

where ρ_0 is the density at time $t = 0$.