Solutions VI.

1 The difference is now we have to add gravity to the pressure gradient in the pipe, i.e. replace $\mathrm{d}P/\mathrm{d}x$ by $\mathrm{d}P/\mathrm{d}x + g$. The rest carries through and the answer is $\nu = 0.1055\ \mathrm{m}^2\ \mathrm{s}^{-1}$.

2 Steady, two-dimensional, incompressible, fully-developed flow. Take the x-axis along the plane and the y-axis normal to the plane. The velocity profile is then $(u(y), 0)$ which satisfies the incompressibility condition. The two components of the Navier–Stokes equations are

$$0 = -\frac{\partial p}{\partial x} + \mu \frac{\mathrm{d}^2 u}{\mathrm{d}y^2} + \rho g \sin\theta, \qquad 0 = -\frac{\partial p}{\partial y} + \rho g \cos\theta,$$

where θ is the angle the plane makes with the horizontal. There is no pressure gradient along the plane, because the pressure at the upper surface is constant. Hence we can integrate the first equation twice:

$$u = -\frac{\rho g \sin\theta y^2}{2\mu} + Ay + B.$$

The two boundary conditions are $u(0) = 0$ and $\mathrm{d}u/\mathrm{d}y = 0$ at $y = h$ (this says that the stress on the upper surface is zero; in fact it is non-zero, but very small and is ignored). Solving for A and B gives

$$u = -\frac{\rho g \sin\theta y(y - 2h)}{2\mu}.$$

3 Steady, two-dimensional flow with no θ-dependence. The only component of velocity is u_θ in the azimuthal direction, which means that the incompressibility condition is satisfied. The θ-component of Navier–Stokes is then (E-5)

$$0 = \frac{\mu}{r}\frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{1}{r}\frac{\mathrm{d}(r u_\theta)}{\mathrm{d}r}\right)$$

(there is no pressure gradient in the azimuthal direction). Integrating twice gives $u_\theta = Ar + B/r$. T1he boundary conditions are $u_\theta = R_1\Omega_1$ at $r = R_1$ and $u_\theta = R_2\Omega_2$ at $r = R_2$. Solving for A and B gives

$$u_\theta = \frac{1}{1 - (R_1/R_2)^2}\left\{\left[\Omega_2 - \Omega_1\left(\frac{R_1}{R_2}\right)^2\right] r + \frac{R_1^2}{r}(\Omega_1 - \Omega_2)\right\}.$$

4 The torque is the integral of force times distance from the axis. The stress is (Exercise 7.14)

$$\tau_{r\theta} = \mu r \frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{u_\theta}{r}\right) = -2\mu \frac{R_1^2}{r}(\Omega_1 - \Omega_2)$$

(notice that the r term in the velocity cancels: rigid body rotation does not contribute to viscous stress). Take the height of the cylinder to be l. The area of the cylinders are $2\pi R_1 l$ and $2\pi R_2 l$. The torque on the first cylinder is $4\pi R_1^2 l \mu(\Omega_1 - \Omega_2)$. The torque on the second cylinder is also $4\pi R_1^2 l \mu(\Omega_1 - \Omega_2)$.

5 Assume inviscid (given steady and axisymmetric). The Euler equations are

$$-\rho \frac{v_\theta^2}{r} = -\frac{\partial p}{\partial r}, \qquad 0 = -\frac{\partial p}{\partial \theta}, \qquad 0 = -\frac{\partial p}{\partial z} - \rho g.$$

Integrating gives

$$\begin{aligned} p &= p_1 - \rho g z + \tfrac{1}{2}\rho\Omega r^2 \qquad \text{for } r \le a, \\ p &= p_2 - \rho g z - \frac{\rho\Omega a^4}{2r^2} \qquad \text{for } r \ge a. \end{aligned}$$

The free surface $z = \zeta$ is at $p = p_a$. Hence

$$\begin{aligned} 0 &= -\rho g(\zeta - \zeta_0) + \tfrac{1}{2}\rho\Omega r^2 \qquad \text{for } r \le a, \\ 0 &= -\rho g(\zeta - \zeta_\infty) - \frac{\Omega a^4}{2r^2} \qquad \text{for } r \ge a. \end{aligned}$$

The pressure is continuous at $r = a$, so $\rho g \zeta_0 + \frac{1}{2}\rho\Omega a^2 = \rho g \zeta_\infty - \frac{1}{2}\Omega a^2$. Hence

$$\zeta_\infty - \zeta_0 = \frac{\Omega a^2}{g}.$$

Steady Bernoulli gives an expression for the Bernoulli function B that is constant along streamlines. Hence it is different for different values of r, since the streamlines are circles. Irrotational Bernoulli is not restricted to streamlines but is only valid for $r > a$.