Solutions IX.

1 Height of the Sears Tower: 442 m. Width: 150 ft $= 45.75$ m (assume uniform square cross-section – not true). Then $h/W = 9.66$. Take $\rho = 1.25\ \mathrm{kg\,m^{-3}}$. Table 9.3 on p. 449 of e.g. Fox & McDonald gives $C_D = 1.2$ (cylinder would probably work OK). Assume $U = 10\ \mathrm{m\,s^{-1}}$. Then $F_D = 1.52 \times 10^6$ N.

2 For small animals, the Reynolds number would be very small due to low velocity flow and small size of the fish. Thus flow would be laminar, and most of the drag force would be due to the viscous forces, resulting in a symmetric flow and no flow separation from the body.

For somewhat larger animals, flow might become unstable at the end of the body since the size of the fish and flow velocity might increase, resulting in higher Reynolds number and flow separation might occur somewhere along the body.

For whales, flow might become turbulent at some point along the body (high Reynolds number due to high velocity and size of the animal), preventing early flow separation and thus reducing drag. For this flow inertial forces would be the major contributor to the drag force and the inertial forces nearly negligible.

The cross-section of the swimming animal is similar to the cross-section of the airplane wing, or airfoil. The airfoil is a streamlined body designed to delay flow separation and thus reduce drag during given flow conditions.

3 Find constants a and b. Boundary conditions: (i) $u = 0$ at $y = 0$: automatic; (ii) $u = U$ at $y = \delta$: $a \sin \delta = U$; (iii) $\mathrm{d}u/\mathrm{d}y = 0$ at $y = \delta$: $ab \cos b\delta = 0$; (iv) (optional) $\mathrm{d}^2u/\mathrm{d}y^2 = 0$ at $y = 0$: automatic. Condition (iii) gives $b\delta = n\pi/2$. The profile with $n = 1$ is the correct one (the others have reversed flow). Then (i) gives $a = U$. This gives $u = U \sin(\pi y/2\delta)$.

Now solve

$$\tau = \mu \left.\frac{\partial u}{\partial y}\right|_{y=0} = \rho U^2 \frac{\mathrm{d}}{\mathrm{d}x} \int_0^\delta \frac{u}{U}\left(1 - \frac{u}{U}\right) \mathrm{d}y.$$

This gives

$$\frac{\pi \mu U}{2\delta} = \int_0^\delta \sin\frac{\pi y}{2\delta}\left(1 - \sin\frac{\pi y}{2\delta}\right) \mathrm{d}y = \frac{\mathrm{d}\delta}{\mathrm{d}x}\frac{4-\pi}{2\pi}.$$

Separate variables and get

$$\delta \mathrm{d}\delta = 11.5 \frac{\mu}{\rho U} \mathrm{d}x;$$

Taking $\delta = 0$ at $x = 0$ gives

$$\frac{\delta}{x} = \frac{4.8}{\sqrt{\mathrm{Re}_x}}.$$

The Blasius result has 5 rather than 4.8.

4 Velocity profile $u = U \sin(\pi y / 2\delta)$. From **3** we have momentum thickness:

$$\Theta = \int_0^\delta \frac{u}{U}\left(1 - \frac{u}{U}\right) \mathrm{d}y = \int_0^\delta \sin\frac{\pi y}{2\delta}\left(1 - \sin\frac{\pi y}{2\delta}\right) \mathrm{d}y = \delta \frac{4-\pi}{2\pi}.$$

Displacement thickness:

$$\delta^* = \int_0^\delta \left(1 - \frac{u}{U}\right) \mathrm{d}y = \int_0^\delta \left(1 - \sin\frac{\pi y}{2\delta}\right) \mathrm{d}y = \delta \frac{\pi - 2}{\pi}.$$

Special problem **1** Should be $\partial_x = 0$. The only relevant component of NS is the x-component. Fully-developed means that the $\mathbf{u}\cdot\nabla u$ terms vanish. There is no pressure gradient either. The only viscous terms that survive are the ∂_y^2 ones, so

$$\frac{\partial u}{\partial t} = \nu \frac{\partial u}{\partial y^2}.$$

Boundary conditions: initial condition of rest, so $u = 0$ at $t = 0$. No-slip at the moving boundary: $u = 1$ at $y = 0$. Flow is at rest far from the plate: $u \to 0$ as $y \to \infty$.

2 Use chain rule:

$$\frac{\partial u}{\partial t} = \frac{\partial \eta}{\partial t}\frac{\mathrm{d}f}{\mathrm{d}\eta} = -\frac{\eta}{2t} f'(\eta)$$

and

$$\frac{\mathrm{d}\partial^2 u}{\mathrm{d}\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial \eta}{\partial y} f'\right) = \frac{\partial}{\partial y}\left(\frac{1}{\sqrt{4\nu t}} f'\right) = \frac{1}{\sqrt{4\nu t}}\frac{\partial \eta}{\partial y} f'' = \frac{f''}{4\nu t}.$$

Substitute into the diffusion equation:

$$-\frac{\eta}{2t} f' = \nu \frac{f''}{4\nu t}.$$

Boundary conditions $y = 0$ corresponds to $\eta = 0$, so there $f = 1$. On the other hand $t = 0$ and $y \to \infty$ both correspond to $\eta \to \infty$, so there $f = 0$. Hence we have

$$f'' + 2\eta f' = 0 \qquad \text{with } f = 1 \text{ at } \eta = 0 \text{ and } f = 0 \text{ as } \eta \to \infty.$$

3 Separate variables or use an integrating factor to get

$$f' = A\mathrm{e}^{-\eta^2}.$$

Integrate once again to get

$$f = A \int_0^\eta \mathrm{e}^{-u^2}\,\mathrm{d}u + B.$$

The boundary condition at the origin gives $B = 1$. The boundary condition as $\eta \to \infty$ gives $A\sqrt{\pi}/2 + B = 1$. Hence

$$f = u/U_p = \mathrm{erfc}(\eta) = 1 - \mathrm{erf}(\eta) = 1 - \frac{2}{\sqrt{\pi}} \int_0^\eta \mathrm{e}^{-u^2}\,\mathrm{d}u.$$

4 Should be $t = x/U$. Calculate wall stress:

$$\tau_w = \mu \left.\frac{\partial u}{\partial y}\right|_{y=0} = U_p \frac{\partial \eta}{\partial y} f'(0) = -\frac{2\mu U_p}{\sqrt{4\pi\nu t}} = -\mu\pi^{-1/2} U_p \sqrt{\frac{U_p}{\nu x}}.$$

Apart from the minus sign, this is the same as (12-30) with $\pi^{-1/2} = 0.5642$ replacing 0.332.

5 We are given that $f(1.82) = 0.01$ so the edge of boundary layer is at $\eta = \delta/\sqrt{4\nu x} = 1.82$. Hence $\delta = 1.82\sqrt{4\nu x} = 3.64\sqrt{\nu x}$. The Blasius result (12-28) is $\delta = 5\sqrt{\nu x}$.