MAE 105 Review

Topis Covered:

Ch 1: Heat Equation

- Derivation of Eq (Physical meaning of each term)
- BCs (physical meaning, zero temp, insulated ends)
- Equilibrium temperature distribution
  - * From Equation, $\frac{\partial u}{\partial t} = 0$
  - * After getting the final solution, $t \to \infty$

Ch 2: Method of Separation of Variables (2.1 - 2.5.2) — Heat Equation, Laplace Equation

$\boxed{\text{Recipe}}$:

1° Separate Eq + BC

2° Solve the Eigenvalue Problem

Sturm-Liouville Theory:
* Comment on λ (R Q)
* Gaurantee the orthogonality relations used

3° Solve the remaining Eq

4° Reunion

5° Superposition e.g. $u(x,t) = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{L} e^{-\left(\frac{n\pi}{L}\right)^2 t}$

6° Apply I.C. — Fourier Series

$u(x,0) = f(x)$

$\Rightarrow \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{L} = f(x)$

Ch3: Fourier Series (3.1 - 3.3)

- Fourier series, Fourier sine series, Fourier cosine series (for $-L < x < L$)
- Jump discontinuity, Fourier series converges to the average of two limits
- Even / Odd Extensions (for $0 < x < L$)

Ch4: Wave Equation (4.1 - 4.5)

- Derivation of Eq (Physical meaning)
- BCs (physical meaning, fixed end, free ends)
- Separation of Variables

Ch5: Sturm-Liouville Theory (5.1 - 5.4, 5.6 - 5.7)

- Rayleigh Quotient for eigenvalues
- Orthogonality relation gauranteed
- Turning non-SL form to S-L form

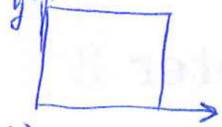
Ch 7 = Higher Dimension PDE (7.1 - 7.3)

→ Vibrating Rectangular Membrane $u(x,y,t)$

X Vibrating Circular Membrane (7.7)

→ Recipe = applies to other similar higher dimension PDE

e.g. Unsteady Heat Conduction over a plate



$$\frac{\partial u}{\partial t} = k\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right)$$

$u(x,y,t)$

Ch 8 = Nonhomogeneous Problems (8.1-8.3)

→ $Q(x)$ and/or Inhomo BCs → Consider $u_E(x)$

→ $Q(x,t)$ → Homo. BCs → Method of Eigenfunction Expansion

→ Inhomo BCs → Consider $r(x,t)$ → Method of Eigenfunction Expansion

(Ch 10 = Infinite Problems = Fourier Transform) ← Not in Final

Partial Differential Equation

Partial Differential Equation $\xrightarrow{\text{infinite domain}}$ Fourier Transform

↓ finite domain

(A) Homogeneous

Two Variables:

* Heat Conduction in a rod:

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2}$$

* Laplace Eq over a plate:

Rectangular:

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

Circular:

$$\frac{1}{r}\frac{\partial}{\partial r}\left[r\frac{\partial u}{\partial r}\right] + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0$$

* Wave Equation (Vibrating String)

$$\frac{\partial^2 u}{\partial t^2} = c^2\frac{\partial^2 u}{\partial x^2}$$

↓

$\boxed{\text{Separation of Variables}}$

1° Sep. of Variables (Eq + BC)

2° Solve the EVP (two homo. BCs)

3° Solve the other ODE

4° Reunion

5° Superposition

6° Apply I.C. (s) / BC

Wave Eq.

Laplace Eq

→ Orthogonality Relations

Three Variables:

* Vibrating Membrane

$$\frac{\partial^2 u}{\partial t^2} = c^2\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right)$$

* Unsteady heat conduction over a plate

$$\frac{\partial u}{\partial t} = c^2\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right)$$

↓

$\boxed{\text{Sep. of Variables twice}}$

1° Sep. of Variables (Eq + BC)

$u(x,y,t) = \phi(x,y)\,h(t)$

2° Solve the PDE

- → 2.1° Sep. of Var.
- → 2.2° Solve the EVP
- → 2.3° Solve the other ODE
- → 2.4° Little Reunion

$\phi(x,y)$ = found!

3° Solve the remaining ODE

$h(t)$ = found!

4° Big Reunion

$u(x,y,t) = \phi(x,y)\,h(t)$

5° Superposition

6° Apply I.C. (s)

Wave Eq.

(B) Inhomogeneous. (Eq + BC)

$Q(x)$

↓

$\boxed{\text{Consider } u_E(x)}$

take care of both the inhomogeneous terms in Eq & BC

1° $u = u_E(x) + v(x,t)$

2° Obtain Eq + BC for $u_E(x)$ & $v(x,t)$

3° Solve for $u_E(x)$ with BCs

4° Solve for $v(x,t)$ using sep. of var.

5° Combine them and apply I.C. for $u(x,t)$

$Q(x,t)$

Homo BCs

↓

$\boxed{\text{Method of Eigenfunction Expansion}}$

1° Obtain $\phi_n(x)$ from a related homogeneous PDE

2° Expand

$$u(x,t) = \sum_{n=1}^{\infty} a_n(t)\,\phi_n(x)$$

Expand

$$Q(x,t) = \sum_{n=1}^{\infty} q_n(t)\,\phi_n(x)$$

3° Plug it back to $eq^{\underline{n}}$ and obtain $eq^{\underline{n}}$ for $a_n(t)$

4° Solve for

$$a_n(t) = a_n^h(t) + a_n^p(t)$$

5° Apply I.C.

Inhomo BCs

↓

$\boxed{\text{Consider a reference temp. } r(x,t)}$

1° $u = r(x,t) + v(x,t)$

Obtain BCs for $r(x,t)$ & $v(x,t)$

2° Pick $r(x,t)$ satisfying BCs:

→ geometrically

→ algebraically

3° Plug back to equation and get a new inhomogeneous PDE for $v(x,t)$ with homogeneous BCs

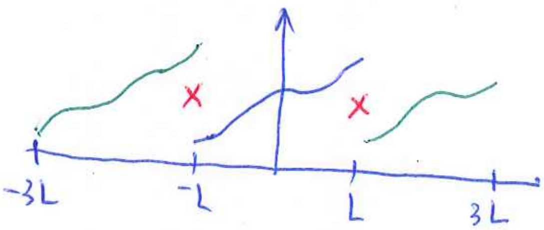
4° Solve for $v(x,t)$ using method of eigenfunction expansion

5° Combine and apply I.C.

Backbone Theories

Fourier Series

* represent the periodic extension of a function defined on $-L < x < L$ by cosines & sines



$$f(x) = A_0 + \sum_{n=1}^{\infty} A_n \cos\frac{n\pi x}{L} + \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{L}$$

* Convergence Theorem:
Fourier series converges to the periodic extension everywhere continuous, and converges to averages of the two limits at jump discontinuities

Prove Even/odd functions

* Sketch of the Fourier series
* Calculate the coefficients (Full information, $-L<x<L$)

Even function

$$f(x) = A_0 + \sum_{n=1}^{\infty} A_n \cos\frac{n\pi x}{L}$$

$$A_0 = \frac{1}{L}\int_0^L f(x)\,dx$$

$$A_n = \frac{2}{L}\int_0^L f(x)\cos\frac{n\pi x}{L}\,dx$$

$$B_n = 0$$

$\boxed{\text{Fourier cosine series}}$

Neither

$$f(x) = A_0 + \sum_{n=1}^{\infty} A_n \cos\frac{n\pi x}{L} + \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{L}$$

$$A_0 = \frac{1}{2L}\int_{-L}^{L} f(x)\,dx$$

$$A_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{n\pi x}{L}\,dx$$

$$B_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{n\pi x}{L}\,dx$$

$\boxed{\text{Fourier series}}$

odd

$$f(x) = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{L}$$

$$A_0 = 0$$

$$A_n = 0$$

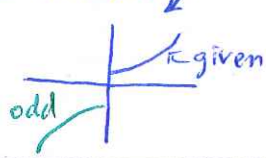
Integration by parts

$$B_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}\,dx$$

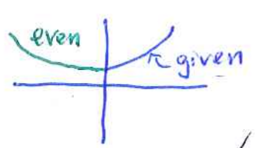
$\boxed{\text{Fourier sine series}}$

* Half Information, $0<x<L$

odd extension



Even extension



Normal function defined on $-L<x<L$, Full information!

Even function → Fourier cosine series

odd function → Fourier sine series

Sturm-Liouville Theory

* S-L form:

$$\frac{d}{dx}\left[p(x)\frac{d\phi}{dx}\right] + q(x)\phi + \lambda\sigma(x)\phi = 0, \quad a<x<b$$

BCs:

$$\beta_1\phi(a) + \beta_2\frac{d\phi}{dx}(a) = 0$$

$$\beta_3\phi(b) + \beta_4\frac{d\phi}{dx}(b) = 0$$

* Rayleigh Quotient:

$$\lambda = \frac{-p\phi\frac{d\phi}{dx}\Big|_a^b + \int_a^b\left[p\left(\frac{d\phi}{dx}\right)^2 - q\phi^2\right]dx}{\int_a^b \phi^2\sigma\,dx}$$

Can $\lambda = 0$?

* Orthogonality Relation Guaranteed

$$\int_a^b \phi_n(x)\phi_m(x)\underbrace{\sigma(x)}_{\text{weight}}\,dx = 0 \quad \text{if } n\neq m$$

* Turn an eq^n of non-SL form to a SL form

1° Make the coefficient of 2nd derivative 1.

e.g. $$x\frac{d^2f}{dx^2} + \frac{df}{dx} + \frac{\lambda}{x}f = 0$$

$$\Rightarrow \frac{d^2f}{dx^2} + \frac{df}{dx} + \frac{\lambda}{x^2}f = 0$$

2° Multiply through by $H(x)$

$$\underbrace{H\frac{d^2f}{dx^2} + \frac{3}{x}H\frac{df}{dx}} + \frac{\lambda}{x}Hf = 0$$

Hope: $$\frac{d}{dx}\left[H\frac{df}{dx}\right] = H\frac{d^2f}{dx^2} + \frac{dH}{dx}\frac{df}{dx}$$

Need: $$\frac{dH}{dx} = \frac{3}{x}H$$

$$\Rightarrow \frac{dH}{H} = \frac{3}{x}dx \Rightarrow \ln H = 3\ln x + c$$

$$\Rightarrow H = e^{3\ln x + c} = De^{3\ln x} = De^{\ln x^3} = Dx^3$$

Pick $D=1 \Rightarrow H = x^3$

3° Multiply through by x^3:

$$x^3\frac{d^2f}{dx^2} + 3x^2\frac{df}{dx} + \lambda x^2 f = 0$$

$$\Rightarrow \boxed{\frac{d}{dx}\left(x^3\frac{df}{dx}\right) + \lambda x^2 f = 0} \text{ SL!}$$