MAE 105 Review Notes

Method of Separation of Variables (MSV)

This method only applies to linear, homogeneous PDEs with linear, homogeneous, boundary conditions. A linear operator, by definition, satisfies:

$$\mathrm{L}(Au_1 + Bu_2) = A\mathrm{L}(u_1) + B\mathrm{L}(u_2)$$

where A and B are arbitrary constants. A linear equation for u is given by

$$\mathrm{L}(u) = f$$

where $f = 0$ for a **homogeneous** equation. As an example, the linear operator for the heat equation is given by

$$\frac{\partial}{\partial t} - k\frac{\partial^2}{\partial x^2}$$

thus giving the linear, homogeneous equation

$$\frac{\partial u}{\partial t} - k\frac{\partial^2 u}{\partial x^2} = 0$$

when it operates on u. The idea behind using MSV is to write u as a product of n functions, where n is the number of variables that describe u, so that each function depends only on one variable. This takes 1 PDE and breaks it up into n ODES. For example, if we are trying to solve the heat equation, the heat, u, is a function of space and time, and is therefore given by $u(x,t)$ in the 1D case. The variables here are x and t, and using MSV, the PDE will become 2 ODEs. In the multidimensional case, $u = u(x,y,t)$, and so the three variables (x,y, and t) will lead to 3 ODES after using MSV.

The Heat Equation

The heat equation with no sources of heat energy and constant thermal coefficients is given by

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2}$$

Let's look at the finite interval case, namely on $0 < x < L$ and $t > 0$. The initial condition is given most generally by $u(x,0) = f(x)$. Note that this is at $t = 0$, and hence is the *initial* condition. The heat equation is a first-order in time, second-order in space PDE, meaning that we require 2 boundary conditions (i.e. in space) and one initial condition (i.e. in time) to close the problem (to solve it fully, such that there are no unknown coefficients). Now let's say that u can be found by assuming a solution that is the product of two functions:

$$u(x,t) = \phi(x)G(t)$$

Therefore, the respective parts of the heat equation become:

$$\frac{\partial u}{\partial t} = \phi(x)\frac{\mathrm{d}G}{\mathrm{d}t}$$

$$k\frac{\partial^2 u}{\partial x^2} = kG(t)\frac{\mathrm{d}^2\phi}{\mathrm{d}x^2}$$

so that

$$\phi(x)\frac{\mathrm{d}G}{\mathrm{d}t} = kG(t)\frac{\mathrm{d}^2\phi}{\mathrm{d}x^2}$$

dividing through by $k\phi(x)G(t)$ gives

$$\frac{1}{kG(t)}\frac{\mathrm{d}G}{\mathrm{d}t} = \frac{1}{\phi(x)}\frac{\mathrm{d}^2\phi}{\mathrm{d}x^2}$$

We will set this equal to a separation constant, $-\lambda$, so that the solution decays in time and oscillates in space (under the assumption that $\lambda > 0$, which we will prove later). Therefore, the time-dependent equation becomes

$$\frac{\mathrm{d}G}{\mathrm{d}t} = -\lambda kG(t)$$

which is a first-order, constant coefficient ODE. There are two ways to solve this: you can separate variables and integrate:

$$\frac{\mathrm{d}G}{G} = -\lambda k\mathrm{d}t$$

$$\ln G = -\lambda kt + A$$

$$G = A\mathrm{e}^{-\lambda kt}$$

or if you prefer to assume the solution as e^{rt} where r corresponds to the roots (as we do for constant coefficient ODEs), then you will find

$$G(t) = \mathrm{e}^{rt}$$

$$\frac{\mathrm{d}G}{\mathrm{d}t} = r\mathrm{e}^{rt}$$

so that the ODE becomes

$$r\mathrm{e}^{rt} = -\lambda k\mathrm{e}^{rt}$$

$$r = -\lambda k$$

and the solution is

$$G(t) = A\mathrm{e}^{rt} = A\mathrm{e}^{-\lambda kt}$$

where A is an arbitrary constant. The boundary value problem (BVP) is given by:

$$\frac{\mathrm{d}^2\phi}{\mathrm{d}x^2} = -\lambda\phi$$

or re-writing this into the form that we are used to seeing,

$$\frac{\mathrm{d}^2\phi}{\mathrm{d}x^2} + \lambda\phi = 0 \tag{1}$$

Now let's think about the various boundary conditions and different signs of λ.

Dirichlet BCs (Ch. 2, pg. 38)

Dirichlet boundary conditions specify the value a solution takes on the boundary of the domain. For example, if we are talking about heat distribution, u, along some rod (so that our domain goes from $0 < x < L$) , Dirichlet conditions would prescribe the temperature at x = 0 and x = L. Homogeneous Dirichlet boundary conditions for this problem are given by:

$$u(0,t) = 0 \quad \& \quad u(L,t) = 0$$

Now recall that we assumed a solution for u of the form $u(x,t) = \phi(x)G(t)$. In order to apply these Dirichlet boundary conditions, we must write them in terms of the function in which we are interested, which is $\phi(x)$ in this case. Therefore, the boundary conditions become:

$$u(0,t) = \phi(0)G(t) = 0 \quad \& \quad u(L,t) = \phi(L)G(t) = 0$$

In order to get nontrivial solutions for u, we cannot have $G(t) = 0$, (otherwise u = 0), therefore we require that

$$\phi(0) = 0 \quad \& \quad \phi(L) = 0$$

Now, there are two ways that you have learned to prove to yourself that $\lambda > 0$ in the case of Dirichlet boundary conditions. The first one is what you did in the beginning of the quarter, which includes solving the boundary value problem (BVP) and taking into account all possible values of λ. Let's revisit that.

Case I: $\lambda = 0$

This will collapse the boundary-value ODE into

$$\frac{\mathrm{d}^2\phi}{\mathrm{d}x^2} = 0$$

and integrating twice, we find that

$$\phi(x) = Ax + B$$

where A and B are arbitrary constants of integration, that we will find by applying the boundary conditions. The two boundary conditions imply the following:

$$\phi(0) = A(0) + B = 0$$

$$B = 0$$

and at $x = L$,

$$\phi(L) = A(L) = 0$$

$$A = 0$$

Therefore, there are no eigenvalues when $\lambda = 0$.

Case II: $\lambda < 0$

Let's recall the full general boundary value ODE,

$$\frac{\mathrm{d}^2\phi}{\mathrm{d}x^2} + \lambda\phi = 0$$

This is a second-order ODE with constant coefficients, and can be solved by assuming a solution of the form $\phi = \mathrm{e}^{rx}$ so that $\frac{\mathrm{d}^2\phi}{\mathrm{d}x^2} = r^2\mathrm{e}^{rx}$. This gives the following:

$$r^2\mathrm{e}^{rx} + \lambda\mathrm{e}^{rx} = 0$$

$$r^2 + \lambda = 0$$
$$r = \pm\sqrt{-\lambda}$$

If $\lambda < 0$, the argument in the square root is real, so that the solution can be written in terms of sinh and cosh:

$$\phi(x) = A\cosh(\sqrt{-\lambda}x) + B\sinh(\sqrt{-\lambda}x)$$

And applying the boundary condition at $x = 0$ gives

$$A\cosh(0) + B\sinh(0) = 0$$

we know that $\sinh(0) = 0$ and $\cosh(0) = 1$, and therefore the boundary condition tells us that $A = 0$. The second condition is

$$B\sinh(\sqrt{-\lambda}L) = 0$$

The only point at which $\sinh(\sqrt{-\lambda}L) = 0$ is at 0, and so we have proved that there are no eigenvalues that can exist for $\lambda < 0$.

Case III: $\lambda > 0$

Re-visiting the roots that we found for the general boundary value problem,

$$r = \pm\sqrt{-\lambda}$$

when $\lambda > 0$, the argument in the square root is negative, and can be written as:

$$r = \pm i\sqrt{\lambda}$$

This gives oscillating solutions:

$$\phi(x) = A\cos(\sqrt{\lambda}x) + B\sin(\sqrt{\lambda}x)$$

Applying the boundary condition at $x = 0$

$$A\cos(0) + B\sin(0) = 0$$

$$A = 0$$

and at $x = L$

$$B\sin(\sqrt{\lambda}L) = 0$$

The values at which $\sin(\sqrt{\lambda}L)$ is 0 occur for $n\pi$, where $n = 1, 2, 3, \ldots$ so that

$$\sqrt{\lambda}L = n\pi$$

$$\lambda = \left(\frac{n\pi}{L}\right)^2 \quad \text{for} \quad n = 1, 2, 3, \ldots$$

Therefore, the only eigenvalues that exist are for $\lambda > 0$ in the case of homogeneous Dirichlet boundary conditions.

Using the Rayleigh Quotient (Ch. 5, pg. 167)

We just discovered that the only values of λ for which eigenvalues exist occur when $\lambda > 0$. It was a bit of a trek to prove this however, and we could instead use the Rayleigh quotient to prove it much quicker. Let's look at the BVP equation again:

$$\frac{\mathrm{d}^2\phi}{\mathrm{d}x^2} + \lambda\phi = 0$$

Recall the form of a general Sturm-Liouville differential equation (Ch. 5, pg. 161):

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(p\frac{\mathrm{d}\phi}{\mathrm{d}x}\right) + q\phi + \lambda\sigma\phi = 0$$

In the case of the heat equation BVP, we see that it can be written in this form in order to pull out the values of p, q, and σ.

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\mathrm{d}\phi}{\mathrm{d}x}\right) + \lambda\phi = 0$$

Upon inspection, we see that $p = 1$, $q = 0$, and $\sigma = 1$ here. The Rayleigh quotient is defined as

$$\lambda = \frac{-p\phi\frac{\mathrm{d}\phi}{\mathrm{d}x}\mid_a^b + \int_a^b \left[p(\frac{\mathrm{d}\phi}{\mathrm{d}x})^2 - q\phi^2\right]\mathrm{d}x}{\int_a^b \phi^2\sigma \ \mathrm{d}x}$$

Working on the interval $0 < x < L$, we have $a = 0$, $b = L$, and in the case of Dirichlet boundary conditions, ϕ vanishes at the boundaries, so that the first term is 0. Substituting in the values of p, q, σ, the Rayleigh quotient reduces to

$$\lambda = \frac{\int_0^L p(\frac{\mathrm{d}\phi}{\mathrm{d}x})^2 \ \mathrm{d}x}{\int_0^L \phi^2\sigma \ \mathrm{d}x} \tag{2}$$

Now we can make the following argument: both the denominator and numerator can only be a positive definite number (since they are squared integrals), and only the numerator can be 0. Therefore, we know that $\lambda \geq 0$. Furthermore, in order to obtain $\lambda = 0$, we would need $\frac{\mathrm{d}\phi}{\mathrm{d}x} = 0$ (so that the numerator equals 0), which would mean that ϕ = constant. However, with Dirichlet boundary conditions, ϕ is 0 at the boundaries, implying that if it is equal to a constant, it would have to be 0 everywhere. Thus, in this case, $\lambda = 0$ would give a trivial solution for ϕ, and now we know that $\lambda > 0$ for Dirichlet BCs.

To recap, we have found that eigenvalues only exist for $\lambda > 0$ in the case of Dirichlet BCs, and the corresponding eigenfunction is

$$\phi(x) = B \sin\left(\frac{n\pi x}{L}\right) \quad \text{for} \quad n = 1, 2, 3, \ldots$$

where B is an arbitrary constant. We now have everything that we need to write a general solution for u, since we have found our two functions, $\phi(x)$ and $G(t)$.

$$u(x,t) = \phi(x)G(t) = C \sin\left(\frac{n\pi x}{L}\right) \mathrm{e}^{-\left(\frac{n\pi}{L}\right)^2 kt} \quad \text{for} \quad n = 1, 2, 3, \ldots$$

where $C = AB$. We can use a summation sign to signify that we have an infinite number of superposed solutions. Thus

$$u(x,t) = \sum_{n=1}^{\infty} C_n \sin\left(\frac{n\pi x}{L}\right) \mathrm{e}^{-\left(\frac{n\pi}{L}\right)^2 kt} \tag{3}$$

and the initial condition can be found at $t = 0$ to be

$$u(x,0) = \sum_{n=1}^{\infty} C_n \sin\left(\frac{n\pi x}{L}\right) = f(x) \tag{4}$$

where $f(x)$ is a general function. Let's explore how to find C_n.

The Initial Value Problem

If the initial condition $f(x)$ happens to look exactly like the form of the solution, then it is very easy to pick out the terms that we need (namely C_n and n) to satisfy the initial condition and thus find the full solution. For example, if the initial condition is

$$u(x,0) = 5 \sin\left(\frac{2\pi x}{L}\right)$$

then we can easily spot that to satisfy $u(x,0) = \sum_{n=1}^{\infty} C_n \sin\left(\frac{n\pi x}{L}\right)$, we would need $C = 5$ and $n = 2$, so that the full solution is

$$u(x,t) = 5 \sin\left(\frac{2\pi x}{L}\right) \mathrm{e}^{-\left(\frac{2\pi}{L}\right)^2 kt}$$

You can check that this in fact satisfies the initial condition by substituting in $t = 0$ and recovering the initial condition. In situations where the initial condition does not look anything like the general solution, we need to be more clever in terms of satisfying it.

Namely, we want to use the orthogonality of sines in this case. We therefore multiply both sides of (4) by $\sin m\pi x/L$ (where m is a dummy variable) and integrate over the domain:

$$\int_0^L f(x)\sin\left(\frac{m\pi x}{L}\right)\mathrm{d}x = \sum_{n=1}^{\infty} C_n \int_0^L \sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{m\pi x}{L}\right)\mathrm{d}x$$

We know that when $m = n$ (Prove these integrals to yourself at least once ...),

$$\int_0^L \sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{m\pi x}{L}\right)\mathrm{d}x = L/2$$

or half the length of the interval, and when $m \neq n$,

$$\int_0^L \sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{m\pi x}{L}\right)\mathrm{d}x = 0$$

Therefore, the only value that remains is $m = n$, allowing us to drop the summation and only keep one of the variables (I choose n) and solve for C_n:

$$C_n = \frac{\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)\mathrm{d}x}{\int_0^L \sin^2\left(\frac{n\pi x}{L}\right)\mathrm{d}x} = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)\mathrm{d}x$$

(Notice how familiar the formula for C_n looks to the case where you must find the unknown coefficients for Sturm-Liouville eigenvalue problems?). We have now fully solved the heat equation (3) for any initial condition, given by $u(x,0) = f(x)$, in the case of Dirichlet BCs.

Neumann BCs (Ch. 2, pg. 59)

Neumann conditions specify the heat flow at the boundaries. For example, at $x = 0$, the BC can be given by

$$-K_0(0)\frac{\partial u}{\partial x}(0,t) = \phi(t)$$

where $\phi(t)$ would be given. In the case of homogenous BCs (also known as insulated ends, i.e. no heat flow out of the rod), we have

$$\frac{\partial u}{\partial x}(0,t) = \frac{\partial u}{\partial x}(L,t) = 0$$

In terms of the BVP (1), these would be given as:

$$\frac{\mathrm{d}\phi}{\mathrm{d}x}(0) = \frac{\mathrm{d}\phi}{\mathrm{d}x}(L) = 0$$

Once again, we must explore all possible values of λ. This time, I will only make the argument that $\lambda \geq 0$ by looking at the Rayleigh quotient (2). In the case of homogeneous Neumann BCs, $\frac{\mathrm{d}\phi}{\mathrm{d}x} = 0$, so that the numerator of (2) can in fact be 0. Therefore, the possible values of λ include the eigenvalue $\lambda = 0$. The easiest way to anticipate this result is by noting that when we get cosine eigenfunctions (which we do in this case, stay tuned. . .), we expect $\lambda = 0$ to be important since $\cos(0) \neq 0$. Let's find the corresponding eigenvalues and eigenfunction in the case of Neumann BCs:

Once again, when $\lambda > 0$, the general solution to the BVP is:

$$\phi(x) = A\cos(\sqrt{\lambda}x) + B\sin(\sqrt{\lambda}x)$$

and

$$\frac{d\phi}{dx} = -A\sqrt{\lambda}\sin(\sqrt{\lambda}x) + B\sqrt{\lambda}\cos(\sqrt{\lambda}x)$$

Applying the boundary condition at $x = 0$,

$$\frac{d\phi}{dx}(0) = -A\sqrt{\lambda}\sin(0) + B\sqrt{\lambda}\cos(0) = 0$$

so that

$$B = 0$$

and at $x = L$

$$\frac{d\phi}{dx}(L) = -A\sqrt{\lambda}\sin(\sqrt{\lambda}L) = 0$$

The values at which $\sin(\sqrt{\lambda}L)$ is 0 occur for $n\pi$, where $n = 1, 2, 3, \ldots$ so that

$$\sqrt{\lambda}L = n\pi$$

$$\lambda = \left(\frac{n\pi}{L}\right)^2 \quad \text{for} \quad n = 1, 2, 3, \ldots$$

and now the eigenfunctions are given by cosines instead of sines:

$$\phi(x) = A\cos\left(\frac{n\pi x}{L}\right) \quad \text{for} \quad n = 1, 2, 3, \ldots$$

When $\lambda = 0$, recall that we have

$$\phi(x) = Ax + B$$

and

$$\frac{d\phi}{dx} = A$$

Therefore, at both boundaries, $\phi(x)$ = constant = B, and so $\lambda = 0$ is an eigenvalue here. At $\lambda = 0$, the time-dependent solution is also constant, since $e^{-\lambda kt} = e^{-(0)kt} = 1$. Therefore, the general solution to $u(x, t)$ is the superposition of $\lambda = 0$ and $\lambda > 0$,

$$u(x, t) = A_0 + \sum_{n=1}^{\infty} A_n \cos\left(\frac{n\pi x}{L}\right) e^{-\left(\frac{n\pi}{L}\right)^2 kt} \tag{5}$$

This may also be written as

$$u(x,t) = \sum_{n=0}^{\infty} A_n \cos\left(\frac{n\pi x}{L}\right) \mathrm{e}^{-\left(\frac{n\pi}{L}\right)^2 kt}$$

since pulling out the $n = 0$ term recovers (5). The initial condition, at $t = 0$, is found to be

$$u(x,0) = \sum_{n=0}^{\infty} A_n \cos\left(\frac{n\pi x}{L}\right) = f(x) \tag{6}$$

where $f(x)$ is a general function. To find A_n, we multiply both sides of (6) by $\cos m\pi x/L$ this time (where m is a dummy variable) and integrate over the domain:

$$\int_0^L f(x) \cos\left(\frac{m\pi x}{L}\right) \mathrm{d}x = \sum_{n=1}^{\infty} A_n \int_0^L \cos\left(\frac{n\pi x}{L}\right) \cos\left(\frac{m\pi x}{L}\right) \mathrm{d}x$$

We know that when $m = n$,

$$\int_0^L \cos\left(\frac{n\pi x}{L}\right) \cos\left(\frac{m\pi x}{L}\right) \mathrm{d}x = L/2$$

or half the length of the interval, and when $m \neq n$,

$$\int_0^L \cos\left(\frac{n\pi x}{L}\right) \cos\left(\frac{m\pi x}{L}\right) \mathrm{d}x = 0$$

Once again, the only value that remains is $m = n$, thus the summation is dropped and we switch to using only one of the variables (I choose n). Solving for A_n:

$$A_n = \frac{\int_0^L f(x) \cos\left(\frac{n\pi x}{L}\right) \mathrm{d}x}{\int_0^L \cos^2\left(\frac{n\pi x}{L}\right) \mathrm{d}x} = \frac{2}{L} \int_0^L f(x) \cos\left(\frac{n\pi x}{L}\right) \mathrm{d}x$$

We also have the additional case when $m = n = 0$, which gives

$$\int_0^L \cos(0)\cos(0)\mathrm{d}x = \int_0^L 1 \ \mathrm{d}x = x \ |_0^L = L$$

and therefore

$$A_0 = \frac{1}{L} \int_0^L f(x)\mathrm{d}x$$

We have now fully solved the heat equation (5) for any initial condition, given by $u(x,0) = f(x)$, in the case of Neumann BCs.

Mixed Neumann-Dirichlet BCs

Let's really quickly go over what eigenfunction and eigenvalues we would get if we looked at mixed boundary conditions. For example, consider the BCs

$$\phi(0) = \frac{\mathrm{d}\phi}{\mathrm{d}x}(L) = 0$$

on the general solution

$$\phi(x) = A\cos(\sqrt{\lambda}x) + B\sin(\sqrt{\lambda}x)$$

The condition at $x = 0$ gives

$$\phi(0) = A\cos(0) + B\sin(0) = 0$$

which shows that

$$A = 0$$

and

$$\frac{d\phi}{dx}(L) = B\sqrt{\lambda}\cos(\sqrt{\lambda}L) = 0$$

The values at which $\cos(\sqrt{\lambda}L)$ is 0 occur for $n\pi + \frac{\pi}{2}$, where $n = 0, 1, 2, \ldots$ (or if you prefer, $n\pi - \frac{\pi}{2}$, where $n = 1, 2, 3, \ldots$), and therefore

$$\sqrt{\lambda}L = \pi\left(n + \frac{1}{2}\right)$$

$$\lambda = \frac{\pi^2}{L^2}\left(n + \frac{1}{2}\right)^2 \quad \text{for} \quad n = 0, 1, 2, \ldots$$

with the corresponding eigenfunction

$$\phi(x) = B_n \sin\left(\frac{\pi x}{L}\left(n + \frac{1}{2}\right)\right) \quad \text{for} \quad n = 0, 1, 2, \ldots$$

Periodic BCs (Ch. 2, pg. 63)

In the case of a thin wire in the shape of a circle, we can investigate periodic boundary conditions working on the interval $-L < x < L$, since we expect continuity in the wire as well as in the heat flux, $\phi(-L) = \phi(L)$ and $\frac{d\phi}{dx}(-L) = \frac{d\phi}{dx}(L)$, as they correspond to the same point on the wire. Applying these boundary conditions to the general solution of the BVP (1):

$$\phi(L) = A\cos(\sqrt{\lambda}L) + B\sin(\sqrt{\lambda}L)$$

$$\phi(-L) = A\cos(-\sqrt{\lambda}L) + B\sin(-\sqrt{\lambda}L) = A\cos(\sqrt{\lambda}L) - B\sin(\sqrt{\lambda}L)$$

Equating the two,

$$A\cos(\sqrt{\lambda}L) + B\sin(\sqrt{\lambda}L) = A\cos(\sqrt{\lambda}L) - B\sin(\sqrt{\lambda}L)$$

$$2B\sin(\sqrt{\lambda}L) = 0$$

The second boundary condition gives

$$\frac{d\phi}{dx}(L) = -A\sqrt{\lambda}\sin(\sqrt{\lambda}L) + B\sqrt{\lambda}\cos(\sqrt{\lambda}L)$$

$$\frac{\mathrm{d}\phi}{\mathrm{d}x}(-L) = -A\sqrt{\lambda}\sin(-\sqrt{\lambda}L) + B\sqrt{\lambda}\cos(-\sqrt{\lambda}L) = A\sqrt{\lambda}\sin(\sqrt{\lambda}L) + B\sqrt{\lambda}\cos(\sqrt{\lambda}L)$$

Once again, equating the two,

$$-A\sqrt{\lambda}\sin(\sqrt{\lambda}L) + B\sqrt{\lambda}\cos(\sqrt{\lambda}L) = A\sqrt{\lambda}\sin(\sqrt{\lambda}L) + B\sqrt{\lambda}\cos(\sqrt{\lambda}L)$$

$$2A\sqrt{\lambda}\sin(\sqrt{\lambda}L) = 0$$

Therefore, both boundary conditions show the same result, namely that in order to get eigenvalues for $\lambda > 0$, we need

$$\sin(\sqrt{\lambda}L) = 0$$

giving eigenvalues

$$\lambda = \left(\frac{n\pi}{L}\right)^2 \quad \text{for} \quad n = 1, 2, 3, \ldots$$

with a corresponding eigenfunction,

$$\phi(x) = a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right) \quad \text{for} \quad n = 1, 2, 3, \ldots$$

Prove to yourself that you also get an eigenvalue for $\lambda = 0$, since there is a cosine eigenfunction term. Thus the general solution $u(x,t)$is given by

$$u(x,t) = a_0 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right) \mathrm{e}^{-\left(\frac{n\pi}{L}\right)^2 kt} + \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right) \mathrm{e}^{-\left(\frac{n\pi}{L}\right)^2 kt}$$

and the initial condition is

$$u(x,0) = f(x) = \sum_{n=0}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right)$$

after absorbing the $n = 0$ term into a_n. In order to find the unknown coefficients, we can first multiply both sides by $\sin(m\pi x/L)$ and integrate over the domain $-L < x < L$

$$\int_{-L}^{L} f(x)\sin\left(\frac{m\pi x}{L}\right)\mathrm{d}x = \sum_{n=0}^{\infty} a_n \int_{-L}^{L} \cos\left(\frac{n\pi x}{L}\right)\sin\left(\frac{m\pi x}{L}\right)\mathrm{d}x + \sum_{n=1}^{\infty} b_n \int_{-L}^{L} \sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{m\pi x}{L}\right)\mathrm{d}x$$

We know that when $m = n$,

$$\int_{-L}^{L} \sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{m\pi x}{L}\right)\mathrm{d}x = L$$

or half the length of the interval, and 0 otherwise. Also,

$$\int_{-L}^{L} \cos\left(\frac{n\pi x}{L}\right)\sin\left(\frac{m\pi x}{L}\right)\mathrm{d}x = 0$$

Since sine is an odd function and cosine is an even function, their product is an odd function. The integral of an odd function over a symmetric interval is 0. Therefore, the only value that remains is $m = n$, for the second term, and thus we can pick out the b_n terms. We can likewise do the same in order to find the a_n terms, specifically by multiplying both sides by $\cos(m\pi x/L)$ and integrating over the domain. Thus we find that

$$a_0 = \frac{1}{2L}\int_{-L}^{L} f(x)\mathrm{d}x$$

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)\mathrm{d}x$$

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right)\mathrm{d}x$$

The Wave Equation (Ch. 4, pg. 142)

The wave equation is given by

$$\frac{\partial^2 u}{\partial t^2} = c^2\frac{\partial^2 u}{\partial x^2} \tag{7}$$

This is second-order in time and second-order in space, therefore we expect 2 initial conditions and 2 boundary conditions in order to find the full solution. We will investigate the wave equation with fixed ends, so that

$$u(0,t) = u(L,t) = 0$$

and initial conditions

$$u(x,0) = f(x), \qquad \frac{\partial u}{\partial t}(x,0) = g(x)$$

Applying MSV, and assuming the solution $u(x,t) = \phi(x)G(t)$,

$$\phi(x)\frac{\mathrm{d}^2 G}{\mathrm{d}t^2} = c^2 G(t)\frac{\mathrm{d}^2\phi}{\mathrm{d}x^2}$$

and dividing by $\phi(x)G(t)c^2$, we obtain 2 ODEs

$$\frac{1}{c^2 G}\frac{\mathrm{d}^2 G}{\mathrm{d}t^2} = \frac{1}{\phi}\frac{\mathrm{d}^2\phi}{\mathrm{d}x^2}$$

with the homogeneous boundary conditions

$$\phi(0) = \phi(L) = 0$$

The motivation behind choosing a sign for the separation constant λ comes from looking at the boundary conditions; we want ϕ to be 0 at both ends, suggesting that it must oscillate in order to satisfy these conditions. Therefore, we choose $-\lambda$,

$$\frac{1}{c^2 G}\frac{\mathrm{d}^2 G}{\mathrm{d}t^2} = \frac{1}{\phi}\frac{\mathrm{d}^2\phi}{\mathrm{d}x^2} = -\lambda$$

This gives the boundary value problem with the same Dirichlet boundary conditions that we saw for the heat equation, so I will go ahead and write down the eigenvalues and corresponding eigenfunction, noting that we found eigenvalues only when $\lambda > 0$ with these boundary conditions:

$$\lambda = \left(\frac{n\pi}{L}\right)^2 \quad \text{for} \quad n = 1, 2, 3, \ldots$$

$$\phi(x) = B \sin\left(\frac{n\pi x}{L}\right) \quad \text{for} \quad n = 1, 2, 3, \ldots$$

Now looking at the time-dependent equation, we have

$$\frac{\mathrm{d}^2 G}{\mathrm{d}t^2} + c^2 \lambda G = 0$$

Notice that this is a second-order, constant coefficient ODE with a general solution of the form

$$G(t) = c_1 \cos(c\sqrt{\lambda}t) + c_2 \sin(c\sqrt{\lambda}t)$$

and substituting in the eigenvalue that we found previously,

$$G(t) = c_1 \cos\left(\frac{n\pi ct}{L}\right) + c_2 \sin\left(\frac{n\pi ct}{L}\right)$$

Therefore, the general solution may be found by taking the product of $\phi(x)$ and $G(t)$ and summing over all possible integer values of n,

$$u(x,t) = \sum_{n=1}^{\infty} \left[A_n \sin\left(\frac{n\pi x}{L}\right) \cos\left(\frac{n\pi ct}{L}\right) + B_n \sin\left(\frac{n\pi x}{L}\right) \sin\left(\frac{n\pi ct}{L}\right) \right]$$

The initial conditions are satisfied by

$$u(x,0) = f(x) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi x}{L}\right)$$

and

$$\frac{\partial u}{\partial t}(x,0) = g(x) = \sum_{n=1}^{\infty} B_n \frac{n\pi c}{L} \sin\left(\frac{n\pi x}{L}\right)$$

In order to find the A_n terms, we multiply both sides by $\sin\left(\frac{m\pi x}{L}\right)$ and integrate over the domain, once again taking advantage of the orthogonality of sines. We therefore find that

$$A_n = \frac{2}{L} \int_0^L f(x) \sin\left(\frac{n\pi x}{L}\right) \mathrm{d}x$$

The same technique is used to find B_n, which is then defined by

$$B_n \frac{n\pi c}{L} = \frac{2}{L} \int_0^L g(x) \sin\left(\frac{n\pi x}{L}\right) \mathrm{d}x$$

and therefore,

$$B_n = \frac{2}{n\pi c} \int_0^L g(x) \sin\left(\frac{n\pi x}{L}\right) \mathrm{d}x$$

Laplace's Equation Inside a Rectangle (Ch. 2, pg. 71)

Laplace's equation in cartesian coordinates is given by

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

Noting that this is second-order in x and second-order in y, we anticipate that we will need 4 boundary conditions to fully close the problem. Consider the rectangle given in Figure 1, on the domain $0 < x < L$ and $0 < y < H$, with four nonhomogeneous boundary conditions,

$$u(0, y) = g_1(y)$$
$$u(L, y) = g_2(y)$$
$$u(x, 0) = f_1(x)$$
$$u(x, H) = f_2(x)$$

When faced with the problem of this type, the idea is to break it up into four problems, with each only having one nonhomogeneous boundary condition. The full solution is then given by a superposition of these four individual solutions

$$u(x, y) = u_1(x, y) + u_2(x, y) + u_3(x, y) + u_4(x, y)$$

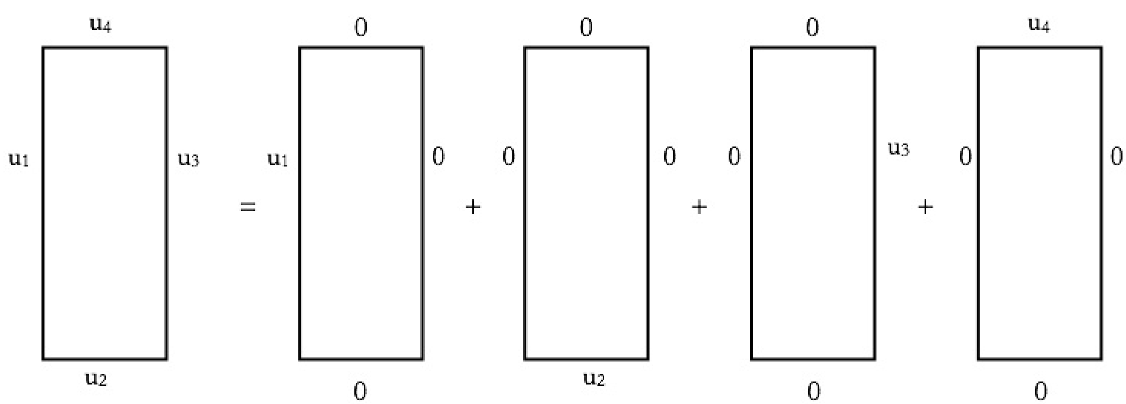

Figure 1: Laplace's equation in a rectangle

Solving for $u_1(x, y)$

The appropriate boundary conditions to solve for $u_1(x, y)$ are

$$u(0, y) = g_1(y)$$

$$u(L, y) = 0$$

$$u(x,0) = 0$$
$$u(x,H) = 0$$

Let's assume that we can write the solution for u as a product of two functions, $u_1(x,y) = h(x)\phi(y)$ so that Laplace's equation can be re-written as

$$\phi(y)\frac{\mathrm{d}^2 h}{\mathrm{d}x^2} + h(x)\frac{\mathrm{d}^2 \phi}{\mathrm{d}y^2} = 0$$

Dividing through by $\phi(y)h(x)$ and bringing the equation for ϕ to the right-hand side, we have

$$\frac{1}{h}\frac{\mathrm{d}^2 h}{\mathrm{d}x^2} = -\frac{1}{\phi}\frac{\mathrm{d}^2 \phi}{\mathrm{d}y^2}$$

Before deciding what sign of λ to choose, let's think about the boundary conditions for this particular problem:

$$u(0,y) = h(0)\phi(y) = g_1(y)$$
$$u(L,y) = h(L)\phi(y) = 0$$
$$u(x,0) = h(x)\phi(0) = 0$$
$$u(x,H) = h(x)\phi(H) = 0$$

The boundary conditions imply that

$$u(0,y) = g_1(y)$$
$$h(L) = 0$$
$$\phi(0) = 0$$
$$\phi(H) = 0$$

Note that we have two homogeneous boundary conditions for $\phi(y)$, and we therefore expect ϕ to oscillate in order to satisfy these conditions. This means that we should set the separation constant to λ, so that the equations look like

$$\frac{1}{h}\frac{\mathrm{d}^2 h}{\mathrm{d}x^2} = -\frac{1}{\phi}\frac{\mathrm{d}^2 \phi}{\mathrm{d}y^2} = \lambda$$

$$\frac{\mathrm{d}^2 \phi}{\mathrm{d}y^2} + \lambda\phi = 0$$

and

$$\frac{\mathrm{d}^2 h}{\mathrm{d}x^2} - \lambda h = 0$$

With Dirichlet boundary conditions for $\phi(y)$, we already know what the eigenvalues and corresponding eigenfunction are:

$$\lambda = \left(\frac{n\pi}{H}\right)^2 \quad \text{for} \quad n = 1, 2, 3, \ldots$$

$$\phi(y) = \sin\left(\frac{n\pi y}{H}\right) \quad \text{for} \quad n = 1, 2, 3, \ldots$$

Also, note that previously when we solved this boundary value problem for the heat equation, we proved that the only eigenvalues that existed in the case of Dirichlet boundary conditions were when $\lambda > 0$. The equation for $h(x)$ has the general solution

$$h(x) = A\sinh(\sqrt{\lambda}x) + B\cosh(\sqrt{\lambda}x)$$

and we have one homogeneous boundary condition at our disposal, $h(L) = 0$. Because this boundary condition is given at $x = L$, it is to our benefit that we shift the origin to $x = L$ in order to get one of the terms in the general solution to disappear after applying the homogeneous boundary condition. Therefore,

$$h(x) = A\sinh\sqrt{\lambda}(x - L) + B\cosh\sqrt{\lambda}(x - L)$$

so that

$$h(L) = A\sinh(0) + B\cosh(0) = 0$$

and therefore $B = 0$. This is allowed because the differential equation does not change upon performing the shift, and is therefore *invariant* upon translation. The solution for $h(x)$ is then

$$h(x) = A\sinh\frac{n\pi}{H}(x - L)$$

where the eigenvalue λ was found when we solved for ϕ. Now the full solution for $u_1(x, y)$ is

$$u_1(x, y) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi y}{H}\right) \sinh\frac{n\pi}{H}(x - L)$$

and the last nonhomogenous boundary condition, $u(0, y) = g_1(y)$ allows us to solve for the unknown coefficients A_n.

$$u_1(0, y) = g_1(y) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi y}{H}\right) \sinh\frac{n\pi}{H}(-L)$$

Once again, multiply both sides by $\sin(m\pi y/H)$ and integrate over the domain so that

$$A_n = \frac{2}{H\sinh\frac{n\pi}{H}(-L)} \int_0^H g_1(y)\sin\left(\frac{n\pi y}{H}\right) \mathrm{d}y$$

There are a few things worth mentioning here. The first is to notice that the hyperbolic term is completely constant in this integral, and that is why we are able to pull it out of the integral and bring it over to the other side. The second point is that the sine function is in terms of y, which is the reason that we are integrating in y from 0 to H, (which is the domain of y here). This is also the reason why we obtain a $2/H$ instead of the usual $2/L$ term in front of the integral to evalue A_n. We now have the full solution for $u_1(x, y)$.

Solving for $u_2(x,y)$

The appropriate boundary conditions to solve for $u_2(x,y)$ are

$$u(0,y) = 0$$

$$u(L,y) = 0$$
$$u(x,0) = f_1(x)$$
$$u(x,H) = 0$$

and writing this in terms of ϕ and h,

$$h(0) = 0$$

$$h(L) = 0$$
$$u(x,0) = f_1(x)$$
$$\phi(H) = 0$$

Now we have two homogeneous boundary conditions for $h(x)$, and therefore expect the solution of $h(x)$ to oscillate in order to satisfy these conditions. This means that we should set the separation constant to $-\lambda$, so that the equations look like

$$\frac{\mathrm{d}^2 h}{\mathrm{d}x^2} + \lambda h = 0$$

$$\frac{\mathrm{d}^2 \phi}{\mathrm{d}y^2} - \lambda \phi = 0$$

and with Dirichlet boundary conditions for $h(x)$, the eigenvalues and corresponding eigenfunction are:

$$\lambda = \left(\frac{n\pi}{L}\right)^2 \quad \text{for} \quad n = 1, 2, 3, \ldots$$
$$h(x) = \sin\left(\frac{n\pi x}{L}\right) \quad \text{for} \quad n = 1, 2, 3, \ldots$$

The equation for $\phi(y)$ has the general solution

$$\phi(y) = A\sinh(\sqrt{\lambda}y) + B\cosh(\sqrt{\lambda}y)$$

with the boundary condition, $\phi(H) = 0$. Making the same argument as before in regards to shifting the origin, application of the homogeneous boundary condition gives

$$\phi(y) = A\sinh\sqrt{\lambda}(y-H) + B\cosh\sqrt{\lambda}(y-H)$$

so that

$$\phi(y) = A\sinh(0) + B\cosh(0) = 0$$

and therefore $B = 0$. The solution for $\phi(y)$ is then

$$\phi(y) = A \sinh \frac{n\pi}{L}(y - H)$$

where the eigenvalue λ was found when solving for $h(x)$. The full solution for $u_2(x, y)$ is

$$u_2(x, y) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{L}\right) \sinh \frac{n\pi}{L}(y - H)$$

and the last nonhomogenous boundary condition, $u(x, 0) = f_1(x)$ allows us to solve for the unknown coefficients B_n.

$$u_2(x, 0) = f_1(x) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{L}\right) \sinh \frac{n\pi}{L}(-H)$$

Multiply both sides by $\sin(m\pi x/L)$ and integrate over the domain in x so that we are left with

$$B_n = \frac{2}{L \sinh \frac{n\pi}{L}(-H)} \int_0^L f_1(x) \sin\left(\frac{n\pi x}{L}\right) \mathrm{d}x$$

I will leave it up to you to find $u_3(x, y)$ and $u_4(x, y)$ Note that with these solutions, you do not have to shift the origin, because the one homogeneous boundary condition will be given at 0.

Laplace's Equation In Polar Coordinates (Ch. 2, pg. 76)

Laplace's equation for a circular disk is given in the book, and I wanted to take the opportunity to solve a similar problem, but on a quarter-circle. Specifically, let's solve

$$\nabla^2 u = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0 \tag{8}$$

on the domain $0 \le \theta \le \pi/2$ and $0 \le r \le 1$ subject to the boundary conditions

$$u(r, 0) = 0, \qquad u(r, \pi/2) = 0, \qquad \frac{\partial u}{\partial r}(1, \theta) = f(\theta)$$

Let's look for solutions of the form $u(r, \theta) = \phi(\theta)G(r)$ so that (8) becomes

$$\phi(\theta)\frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}\left(r\frac{\mathrm{d}G}{\mathrm{d}r}\right) + \frac{1}{r^2}G(r)\frac{\mathrm{d}^2\phi}{\mathrm{d}\theta^2} = 0$$

Dividing through by ϕG and multiplying by r^2, we obtain

$$\frac{r}{G}\frac{\mathrm{d}}{\mathrm{d}r}\left(r\frac{\mathrm{d}G}{\mathrm{d}r}\right) = -\frac{1}{\phi}\frac{\mathrm{d}^2\phi}{\mathrm{d}\theta^2} = \lambda$$

where we have chosen a positive λ, because we expect oscillations in ϕ due to the two homogeneous boundary conditions

$$\phi(0) = \phi(\pi/2) = 0$$

Let's apply these boundary conditions to the general solution

$$\phi(\theta) = A\sin(\sqrt{\lambda}\theta) + B\cos(\sqrt{\lambda}\theta)$$

$$\phi(0) = A\sin(0) + B\cos(0) = 0$$

and therefore $B = 0$. The second boundary condition gives

$$\phi(\pi/2) = A\sin(\frac{\pi}{2}\sqrt{\lambda}) = 0$$

When does the argument of sine $= 0$?

$$\frac{\pi}{2}\sqrt{\lambda} = n\pi \quad \text{for} \quad n = 1, 2, 3, \ldots$$

Solving for λ gives

$$\lambda = (2n)^2 \quad \text{for} \quad n = 1, 2, 3, \ldots$$

and the eigenfunction is given by

$$\phi(\theta) = \sin(2n\theta) \quad \text{for} \quad n = 1, 2, 3, \ldots$$

Now let's look at the r-equation since we have found our eigenvalues. The equation can be re-wrriten so that it is in equidimensional form (Ch. 2, pg. 78)

$$\frac{r}{G}\frac{\mathrm{d}}{\mathrm{d}r}\left(r\frac{\mathrm{d}G}{\mathrm{d}r}\right) = \lambda = (2n)^2$$

$$r^2\frac{\mathrm{d}^2G}{\mathrm{d}r^2} + r\frac{\mathrm{d}G}{\mathrm{d}r} - 4n^2G = 0$$

We therefore seek power solutions, $G = r^p$, consequently leading to the characteristic polynomial

$$p^2 - 4n^2 = 0$$

and solving for p,

$$p = \pm 2n$$

The general solution is thus $G = c_1r^{2n} + c_2r^{-2n}$. In this problem, because the origin is part of the domain (e.g. if this were an annulus, the origin would not be part of the domain), then one requirement is for the solution to stay bounded at the origin, $\mid G(0) \mid < \infty$, which can only occur if c_2 is 0. Therefore, the solution to $G(r)$ is

$$G = c_1r^{2n}$$

and the general solution to $u(r,\theta)$ is given by

$$u(r,\theta) = \sum_{n=1}^{\infty} A_n r^{2n} \sin(2n\theta)$$

There is still one nonhomogeneous boundary condition to apply, $\frac{\partial u}{\partial r}(1,\theta)$. Therefore,

$$\frac{\partial u}{\partial r}(1,\theta) = f(\theta) = \sum_{n=1}^{\infty} 2nA_n 1^{2n-1} \sin(2n\theta)$$

In order to find A_n, multiply both sides by $\sin(2m\theta)$ and integrate from 0 to $\pi/2$. We expect the sine orthogonality integral to return half of the interval (please prove this to yourself), and therefore

$$\int_0^{\pi/2} \sin^2(2n\theta)\mathrm{d}\theta = \pi/4$$

so that

$$A_n = \frac{2}{n\pi} \int_0^{\pi/2} f(\theta) \sin(2n\theta)\mathrm{d}\theta$$

Another interesting problem to consider is solving the Laplace equation in polar coordinates **outside** of a circular disk. Let's quickly explore that with the boundary condition $u(a,\theta) = f(\theta)$. Please find on pgs. 78-79 how to obtain eigenvalues and the corresponding eigenfunction in the case of periodic boundary conditions,

$$\phi(\theta) = \cos(n\theta) + \sin(n\theta) \quad \text{for} \quad n = 0, 1, 2, \ldots$$

The full equation for $G(r)$ after also taking into consideration the $\lambda = 0$ eigenvalue is

$$G(r) = c_1 r^n + d_1 r^{-n} + c_0 + d_0 \ln(r)$$

If we are trying to solve Laplace's equation outside of the circle, we would not expect to worry about solutions blowing up at the origin, since that is not our area of interest. Instead, we would expect $u \to 0$ as $r \to \infty$ and therefore c_1 and d_0 must be 0, otherwise the solution would not decay at ∞. Therefore,

$$G = c_0 + d_1 r^{-n}$$

so that the general solution takes the form

$$u(r,\theta) = \sum_{n=0}^{\infty} A_n r^{-n} \cos(n\theta) + \sum_{n=1}^{\infty} B_n r^{-n} \sin(n\theta)$$

where the c_0 is recovered by substituting $n = 0$ into the general solution, and has thus been absorbed in the cosine summation. The book can be consulted to evaluate A_n and B_n.

Sturm-Liouville Eigenvalue Problems (Ch.5, pg. 161)

The regular Sturm-Liouville differential equation is given by

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(p(x)\frac{\mathrm{d}\phi}{\mathrm{d}x}\right)+q(x)\phi+\lambda\sigma(x)\phi=0$$

on the interval $a<x<b$. Some things to note:

- Any second-order, linear ODE may be written in this form by use of an integrating factor
- All eigenvalues λ are real, and there are an infinite number of eigenvalues
- Corresponding to each eigenvalue λ_n, there is an eigenfunction $\phi_n(x)$ that has $n-1$ zeros on the interval $a<x<b$
- Eigenfunctions are orthogonal relative to the weight function $\sigma(x)$, and the orthogonality relation is given by

$$\int_a^b \phi_n(x)\phi_m(x)\sigma(x)\mathrm{d}x=0 \quad \text{when} \quad n\neq m$$

- Any piecewise smooth function $f(x)$ can be represented by a generalized Fourier series of the eigenfunctions:

$$f(x)\sim\sum_{n=1}^{\infty}a_n\phi_n(x)$$

  If we would like to solve for a_n, we can do as before and multiply both sides by the eigenfunction (using the dummy variable, m) and integrate over the domain. From the previous theorem, we know that the eigenfunctions are orthogonal, therefore only the $n=m$ term will survive and

$$a_n=\frac{\int_a^b f(x)\phi_n(x)\sigma(x)\mathrm{d}x}{\int_a^b \phi_n^2(x)\sigma(x)\mathrm{d}x}$$

Let's look at the most general second-order ODE that we can come up with,

$$\frac{\mathrm{d}^2\phi}{\mathrm{d}x^2}+\alpha(x)\frac{\mathrm{d}\phi}{\mathrm{d}x}+[\lambda\beta(x)+\gamma(x)]\phi=0$$

Multiply this by some function $H(x)$

$$H\frac{\mathrm{d}^2\phi}{\mathrm{d}x^2}+\alpha(x)H\frac{\mathrm{d}\phi}{\mathrm{d}x}+[\lambda\beta(x)+\gamma(x)]\phi H=0$$

and now let $\alpha(x)H$= dH/dx in order to create a perfect derivative in the first term,

$$H\frac{\mathrm{d}^2\phi}{\mathrm{d}x^2} + \frac{\mathrm{d}H}{\mathrm{d}x} + [\lambda\beta(x) + \gamma(x)]\phi H = 0$$

This allows the equation to be written in SL form,

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(H\frac{\mathrm{d}\phi}{\mathrm{d}x}\right) + [\lambda\beta(x) + \gamma(x)]\phi H = 0$$

The only thing left to find is what $H(x)$ is; don't forget that we let $\alpha(x)H$= dH/dx . We can readily integrate this equation to give the integrating factor,

$$\frac{\mathrm{d}H}{\mathrm{d}x} = \alpha(x)H$$

$$\frac{\mathrm{d}H}{H} = \alpha(x)\ \mathrm{d}x$$

$$\ln(H) = \int^x \alpha(x')\mathrm{d}x'$$

$$H = \mathrm{e}^{\int^x \alpha(x')\mathrm{d}x'}$$

Thus we can find $H(x)$ if $\alpha(x)$ is known, and thus we have successfully re-written the second-order general ODE into SL form, where

$$p(x) = H, \quad q(x) = \gamma(x)H, \quad \sigma(x) = \beta(x)H$$

Green's function (Ch. 9, pg. 393)

The Green's function is another tool for solving nonhomogeneous problems subject to two homogeneous boundary conditions. If we are given a nonhomogeneous problem

$$L(u) = f(x)$$

where L is a linear operator acting on u, the solution may be found by seeking the Green's function, since

$$u(x) = \int_a^b f(x_0)G(x, x_0)\mathrm{d}x_0$$

The Green's function $G(x, x_0)$, is the response at x due to a concentrated source at x_0. We will do an example to illustrate how the Green's function works.

Green's function Example

Let's solve the ODE

$$y'' + 3y' + 2y = 0 \quad \text{with} \quad y(0) = y'(0) = 0$$

using Green's functions (the prime denotes derivatives in x). The first step is to apply the linear operator to the Green's function $G(x, x_0)$ and set it equal to the delta function, noting that it must satisfy the same boundary conditions:

1. $$G'' + 3G' + 2G = \delta(x - x_0) \quad \text{with} \quad G(0, x_0) = G'(0, x_0) = 0$$
   When $x \neq x_0, G'' + 3G' + 2G = 0$.

2. Now solve this equation twice, once to the left of the source x_0, and once to the right. This is a 2nd order constant coefficient ODE with the characteristic polynomial:
   $$r^2 + 3r + 2 = 0$$
   $$(r + 2)(r + 1) = 0$$
   $$r = -2, -1$$
   so that
   $$G(x, x_0) = \begin{Bmatrix} Ae^{-2x} + Be^{-x} & x < x_0 \\ Ce^{-2x} + De^{-x} & x > x_0 \end{Bmatrix}$$

3. Now we apply our boundary conditions to evalue 2/4 of the unknown coefficients. The two boundary conditions will be applied to the interval $x < x_0$ in this special situation, and we find that they imply that $A = B = 0$. Therefore,
   $$G(x, x_0) = \begin{Bmatrix} 0 & x < x_0 \\ Ce^{-2x} + De^{-x} & x > x_0 \end{Bmatrix}$$

4. There are still two more conditions to satisfy, namely continuity of G at $x = x_0$, and a jump in the $n - 1$ derivative at $x = x_0$. The former gives
   $$Ce^{-2x_0} + De^{-x_0} = 0$$
   $$C = -De^{x_0}$$

5. The jump condition comes from integrating the original equation and noting that the integral of the delta function is 1.
   $$G' \Big|_{x_0^-}^{x_0^+} = 1$$
   where the other terms equated to 0 since they are continuous across $x = x_0$. Therefore, we have
   $$(-2Ce^{-2x_0} - De^{-x_0}) - 0 = 1$$

and solving the two equations simultaneously, we find that

$$D = e^{x_0}$$

$$C = -e^{2x_0}$$

so that

$$G(x, x_0) = \begin{Bmatrix} 0 & x < x_0 \\ -e^{2x_0}e^{-2x} + e^{x_0}e^{-x} & x > x_0 \end{Bmatrix}$$

6. Therefore, the solution is given by

$$y(x) = \int_0^{x_0} (-e^{2(x_0-x)} + e^{(x_0-x)} f(x_0) dx_0$$

This example is a bit weird in the sense that on the left side of x_0, there is no Green's function, but the symmetry to the right of x_0 can still be seen and exploited in order to write the solution. One thing to note is that for higher-order derivatives (e.g. for a derivative of order n), the Green's function will always have a jump in the $n - 1$ derivative, and will be continuous in all lower derivatives.

Fourier Transforms (Ch. 10, pg. 445)

In order to illustrate how to use Fourier Transforms, I will do an example from the book given on pg. 469. Let's solve the diffusion equation with convection:

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2} + c\frac{\partial u}{\partial x}$$

on the interval $-\infty < x < \infty$ with the initial condition $u(x, 0) = f(x)$. Taking the Fourier transform of this equation, we get

$$\frac{\partial \bar{U}}{\partial t} = -\omega^2 k\bar{U} - ic\omega\bar{U}$$

which can be integrated

$$\frac{\partial \bar{U}}{\partial t} = \bar{U}(-\omega^2 k - ic\omega)$$

$$\frac{\partial \bar{U}}{\bar{U}} = (-\omega^2 k - ic\omega)\partial t$$

$$\bar{U} = A(\omega)e^{(-\omega^2 k - ic\omega)t}$$

In order to find the solution $u(x, y)$, we need to find the inverse Fourier transform of $\bar{U}(\omega, t)$. This can be done by noting that

$$F^{-1}[\bar{U}] = F^{-1}[A(\omega)] * F^{-1}[e^{(-\omega^2 k - ic\omega)t}]$$

where we can use the convolution theorem in order to find the inverse Fourier transform of the product of two functions. Prove to yourself that

$$F^{-1}[A(\omega)] = f(x)$$

which is the initial condition. Now for the second term,

$$F^{-1}[\mathrm{e}^{(-\omega^2 k - ic\omega)t}] = \int_{-\infty}^{\infty} \mathrm{e}^{(-\omega^2 k - ic\omega)t} \mathrm{e}^{-i\omega x} \mathrm{d}\omega$$

The important thing to note here is that this may be re-written as

$$\int_{-\infty}^{\infty} \mathrm{e}^{(-\omega^2 k - ic\omega)t} \mathrm{e}^{-i\omega x} \mathrm{d}\omega = \int_{-\infty}^{\infty} \mathrm{e}^{-\omega^2 kt} \mathrm{e}^{-i\omega(x+ct)} \mathrm{d}\omega$$

In essence, we can introduce a new definition for x, such that $x' = x + ct$. Therefore,

$$u(x+ct, t) = \int_{-\infty}^{\infty} \bar{U}(\omega, t) \mathrm{e}^{-i\omega(x+ct)} \mathrm{d}\omega = \int_{-\infty}^{\infty} \bar{U}(\omega, t) \mathrm{e}^{-i\omega x'} \mathrm{d}\omega$$

We can now find the Fourier transform of above function, since we know from tables that it will be a Gaussian. This gives us

$$\int_{-\infty}^{\infty} \mathrm{e}^{-\omega^2 kt} \mathrm{e}^{-i\omega x'} \mathrm{d}\omega = \sqrt{\frac{\pi}{kt}} \mathrm{e}^{-x'^2/4kt} \sqrt{\frac{\pi}{kt}} \mathrm{e}^{-(x+ct)^2/4kt}$$

All that is left to do is apply the convolution theorem, which states that

$$f * g = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(\bar{x}) g(x - \bar{x}) \mathrm{d}\bar{x}$$

In this case,

$$f(x) = f(x)$$

or the initial condition, and

$$g(x) = \sqrt{\frac{\pi}{kt}} \mathrm{e}^{-(x+ct)^2/4kt}$$

so that

$$u(x,t) = f * g = \frac{1}{\sqrt{4\pi kt}} \int_{-\infty}^{\infty} f(\bar{x}) \mathrm{e}^{-(x+ct-\bar{x})^2/4kt} \mathrm{d}\bar{x}$$