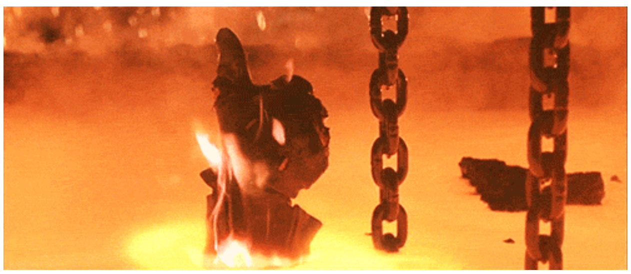
Final Exam

This is a three-hour examination. You may use two hand-written sheets of paper with notes that you have prepared. The six questions each carry the same number of points.

1 Saving the Terminator from a pool of molten metal (10 points). You are interested in rescuing the Terminator from drowning in a pool of molten metal when only his arm is sticking above the pool. A sketch of this scenario is given below.



We model the temperature distribution in Terminator's arm using the one-dimensional heat equation. Assume the length of the arm above the molten metal has length L. The heat equation is

$$\frac{\partial u}{\partial t} = \kappa \frac{\partial^2 u}{\partial x^2}\,.$$

Let's use a constant temperature boundary condition for the extremity of the arm touching the molten metal, so that

$$u(0,t) = u_{\text{metal}}\,,$$

where u_{metal} is the temperature of the pool of molten metal. We model the other side of his arm with the Newton cooling boundary condition, so that

$$-\kappa \frac{\partial u}{\partial x}(L,t) = H\,(u - u_{\text{air}}) \qquad \text{at} \qquad x = L\,,$$

Answer the following:

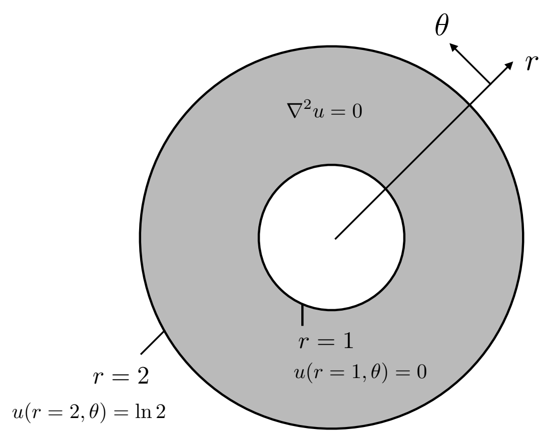
1. What are the units of κ and H? Assume u is in K, t is in seconds, and x is in meters. Explain physically why $H > 0$.
2. Assume that $\partial u/\partial t = 0$ (the Terminator's temperature is not changing in time). What is the temperature distribution $u(x)$ in the Terminator's arm?
3. As the Terminator sinks into the pool, L decreases. Does the decrease in L increase or decrease the temperature at his fingertips $u(x = L)$?

2 Laplace in an annulus (10 points). Consider Laplace's equation in an annulus. The annulus has an inner radius of 1 and and outer radius of 2; a sketch of the annulus domain is given below. Laplace's equation in polar coordinates is

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0\,.$$

The boundary conditions are

$$u(1,\theta) = 0 \qquad \text{and} \qquad u(2,\theta) = \ln 2\,.$$



Answer the following:

(a) Use separation of variables to find the general solution $u(r,\theta)$.

(b) Find the solution which satisfies the boundary conditions at $r = 1$ and $r = 2$.

(c) If u is temperature, the total heat flux flowing *toward the origin* at radius r is

$$Q(r) = \int_0^{2\pi} q(r,\theta) r \, \mathrm{d}\theta\,,$$

where

$$q(r,\theta) = k\frac{\partial u}{\partial r}$$

is the inward heat flux density (the flux in the negative r-direction). Determine $Q(r = 2)$ and $Q(r = 1)$. What do you observe? How could you have predicted this from the governing problem without computing the integrals?

3 Fourier series (10 points).

(a) Find the Fourier sine series of

$$f(x) = \mathrm{e}^x\,,$$

on the interval $0 \le x \le L$.

(b) Find the Fourier cosine series of $f(x)$ on the same interval.

Hint: consider the real and imaginary parts of the integral $\int_0^L \mathrm{e}^{x+\mathrm{i}n\pi x/L}\,\mathrm{d}x$.

4 Green's functions (10 points). Consider steady-state heat conduction in a rod with an arbitrary source $Q(x)$, non-uniform cross-section, and length L. At $x = 0$ the rod is insulated, so there is zero heat flux out of the rod at $x = 0$. At $x = L$ the rod is held at constant temperature.

(a) Explain why the governing equation is

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(A\frac{\mathrm{d}u}{\mathrm{d}x}\right) + Q = 0.$$

(b) The rod is insulated at $x = 0$ such that the heat flux is zero there. What is the corresponding mathematical boundary condition on u at $x = 0$?

(c) Explain why we are mathematically justified in taking $u = 0$ at the end $x = L$, rather than requiring the actual value of the temperature there.

(d) Find the Green's function for the above inhomogeneous equation with $A = (2L - x)^{-1}$.

(e) Express the solution a sum of two integrals. [*Hint: be very clear about which variable is the variable of integration as well as its range in each integral.*]

5 Fourier transform: wave equation with damping (10 points). Consider the equation

$$\frac{\partial^2 u}{\partial t^2} + 2\gamma\frac{\partial u}{\partial t} = c^2\frac{\partial^2 u}{\partial x^2}$$

with initial conditions

$$u(x,0) = f(x) \qquad \text{and} \qquad \frac{\partial u}{\partial t}(x,0) = 0\,.$$

(a) What are the units of γ? What sign do you expect γ to have for well-behaved solutions? [*Hint: You can ignore the right-hand side to answer this.*]

(b) Obtain the equation for the Fourier transform $U(\omega, t)$ of $u(x, t)$. [*Hint: you will find that both solutions for U have the form $U \sim \mathrm{e}^{\alpha t}$. To save yourself some ink, write α as $\alpha = -\gamma \pm \mathrm{i}\sigma$, and give the form of σ in terms of c, γ, and ω.*]

(c) Let $f(x) = \delta(x)$ and find the physical space solution $u(x, t)$. You may leave your answer as an integral.

(d) Define the physical space solution found in (c) as $G(x, t)$. Write the general solution for $u(x, t)$ as a convolution integral involving $G(x, t)$ and an arbitrary initial condition $f(x)$.

6 Using D'Alembert's solution (10 points). Consider the wave equation in an infinite domain:

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}, \qquad \text{where} \qquad -\infty < x < \infty,$$

and the initial conditions on $u(x,t)$ are

$$u(x,0) = 0 \qquad \text{and} \qquad \frac{\partial u}{\partial t}(x,0) = g(x).$$

1. Write down the general solution $u(x,t)$ for arbitrary $g(x)$.
2. Compute and draw the solution at *two* later times when

$$g(x) = x\,\mathrm{e}^{-x^2/2}.$$

[*Hint: because the function* $x\,\mathrm{e}^{-x^2/2}$ *is an exact derivative, it can be integrated in closed form.*]