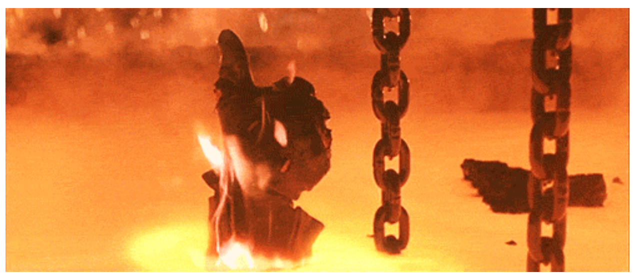
MAE105 Introduction to Mathematical Physics Spring Quarter 2015
http://web.eng.ucsd.edu/~sgls/MAE105_2015/

Final Exam

This is a three-hour examination. You may use two hand-written sheets of paper with notes that you have prepared. The six questions each carry the same number of points.

1 Saving the Terminator from a pool of molten metal (10 points). You are interested in rescuing the Terminator from drowning in a pool of molten metal when only his arm is sticking above the pool. A sketch of this scenario is given below.



We model the temperature distribution in Terminator's arm using the one-dimensional heat equation. Assume the length of the arm above the molten metal has length L. The heat equation is

$$\frac{\partial u}{\partial t} = \kappa \frac{\partial^2 u}{\partial x^2}\,.$$

Let's use a constant temperature boundary condition for the extremity of the arm touching the molten metal, so that

$$u(0,t) = u_{\text{metal}}\,,$$

where u_{metal} is the temperature of the pool of molten metal. We model the other side of his arm with the Newton cooling boundary condition, so that

$$-\kappa \frac{\partial u}{\partial x}(L,t) = H\,(u - u_{\text{air}}) \qquad \text{at} \qquad x = L\,,$$

Answer the following:

1. What are the units of κ and H? Assume u is in K, t is in seconds, and x is in meters. Explain physically why $H > 0$.
2. Assume that $\partial u/\partial t = 0$ (the Terminator's temperature is not changing in time). What is the temperature distribution $u(x)$ in the Terminator's arm?
3. As the Terminator sinks into the pool, L decreases. Does the decrease in L increase or decrease the temperature at his fingertips $u(x = L)$?

Solution.

1. The units of κ are
$$[\kappa] = \left[\frac{\frac{\partial u}{\partial t}}{\partial^2 u}\partial x^2\right] = \frac{\text{m}^2}{\text{s}}\,.$$
The units of H are
$$\left[\frac{\kappa\frac{\partial u}{\partial x}}{u}\right] = \frac{\text{m}^2\text{s}^{-1}\text{Km}^{-1}}{\text{K}} = \frac{\text{m}}{\text{s}}\,.$$

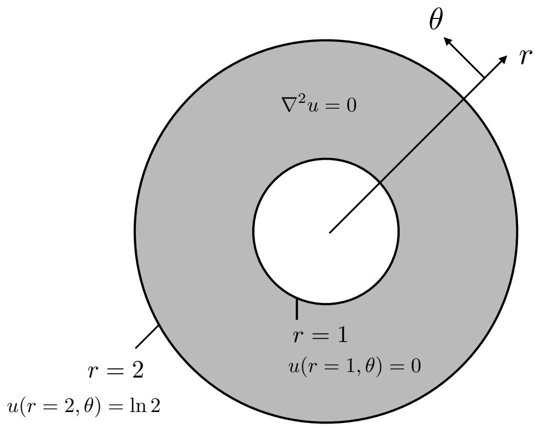
2. The temperature distribution is found by solving
$$\frac{\partial^2 u}{\partial x^2} = 0\,,$$
which yields
$$u = ax + b\,.$$
Satisfying $u = u_{\text{metal}}$ at $x = 0$ implies $b = u_{\text{metal}}$, and the condition at $x = L$ implies
$$-ka = H(aL + u_{\text{metal}} - u_{\text{air}}) \quad \Longrightarrow \quad a = -H\frac{u_{\text{metal}} - u_{\text{air}}}{k + HL} = -\frac{u_{\text{metal}} - u_{\text{air}}}{k/H + L}$$
The solution is
$$u = -\frac{u_{\text{metal}} - u_{\text{air}}}{k/H + L}x + u_{\text{metal}}$$

3. The temperature increases.

2 Laplace in an annulus (10 points). Consider Laplace's equation in an annulus. The annulus has an inner radius of 1 and and outer radius of 2; a sketch of the annulus domain is given below. Laplace's equation in polar coordinates is
$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0\,.$$
The boundary conditions are
$$u(r = 1, \theta) = 0 \qquad \text{and} \qquad u(r = 2, \theta) = \ln 2\,.$$
Answer the following:

(a) Use separation of variables to find the general solution $u(r, \theta)$.

(b) Find the solution which satisfies the boundary conditions at $r = 1$ and $r = 2$.



(c) If u is temperature, the total heat flux flowing *toward the origin* at radius r is

$$Q(r) = \int_0^{2\pi} q(r,\theta) r \, \mathrm{d}\theta \,,$$

where

$$q(r,\theta) = k \frac{\partial u}{\partial r}$$

is the inward heat flux density (the flux in the negative r-direction). Determine $Q(r = 2)$ and $Q(r = 1)$. What do you observe? How could you have predicted this from the governing problem without computing the integrals?

Solution.

(a) To separate variables we propose $u(r,\theta) = f(r)g(\theta)$ and plug this into Laplace's equation. This yields

$$\frac{g}{f}\left(r f'\right)' + \frac{f}{r^2} g'' = 0\,.$$

To isolate terms dependent on r and θ respectively, we multiply by r^2/fg and move the g-terms to the other side of the equation. This yields

$$\frac{r}{f}\left(r f'\right)' = -\frac{g''}{g} = \lambda\,,$$

where we have defined a separation constant λ. The θ-equation is

$$g'' + \lambda g = 0\,.$$

$g(\theta)$ must be periodic such that $g(0) = g(2\pi)$ and $g'(0) = g'(2\pi)$. The solution is therefore

$$g = A \sin(n\theta) + B \cos(n\theta)\,,$$

where we have determined the eigenvalue $n = \sqrt{\lambda} = 0, 1, 2, 3....$ Here, we choose $n \geq 0$ for simplicity; we must choose either $n \geq 0$ or $n \leq 0$. The r-equation is then

$$r^2 f'' + r f' - n^2 f = 0\,.$$

To solve this for $n > 0$, we propose $f_n = Cr^\alpha$, to yield an equation for α,

$$\alpha^2 = n^2\,,$$

which implies $\alpha = \pm n$. Thus $f_n(r)$ is

$$f_n(r) = Cr^n + Dr^{-n}\,.$$

The condition at $r = 1$ implies that $C + D = 0$, or that $D = -C$, and

$$f_n(r) = C\left(r^n - r^{-n}\right)\,.$$

When $n = 0$, this form only gives one of the solutions for $f(r)$. To find the other solution, we return to the r-equation for $n = 0$,

$$r^2 f_0'' + r f_0' = 0\,.$$

This is a first-order equation for f', whose solution is $f_0' = E/r$. We can thus integrate to find f_0:

$$f_0(r) = E \ln r + F\,.$$

The condition at $r = 1$ implies that $F = 0$. Putting f and g together, and adding all the solutions for every n yields

$$u(r, \theta) = E \ln r + \sum_{n=1}^{\infty} \left(r^n - r^{-n}\right)\left[A_n \sin(n\theta) + B_n \cos(n\theta)\right]\,.$$

(b) The boundary condition at $r = 2$ is $u(r, \theta) = \ln 2$. Applying the boundary condition implies that

$$\ln 2 = E \ln 2 + \sum_{n=1}^{\infty} \left(2^n - 2^{-n}\right)\left[A_n \sin(n\theta) + B_n \cos(n\theta)\right]\,.$$

First, we simply integrate this condition from $\theta = 0$ to $\theta = 2\pi$. This eliminates all the terms inside the summation and yields

$$\int_0^{2\pi} \ln 2 \, \mathrm{d}\theta = \int_0^{2\pi} E \ln 2 \, \mathrm{d}\theta\,, \qquad \text{which implies} \qquad E = 1\,.$$

Now, if we multiply by either $\sin(m\theta)$ or $\cos(m\theta)$, the boundary condition on the left side disappears. Thus the solution is just

$$u(r, \theta) = \ln r\,.$$

This also follows from the fact that $u = \ln r$ satisfies the boundary conditions at $r = 1$ and $r = 2$ as well as the governing equation, and so must be the full solution.

(c) The origin-flowing heat flux density is

$$q(r, \theta) = k\frac{\partial u}{\partial r} = \frac{1}{r}\,.$$

Therefore, the total origin-flowing heat flux at $r = 1$ is

$$Q(1) = \int_0^{2\pi} k\,\mathrm{d}\theta = 2k\pi\,.$$

The total origin-flowing heat flux at $r = 2$, on the other hand, is

$$Q(2) = \int_0^{2\pi} \frac{k}{2} 2\,\mathrm{d}\theta = 2k\pi\,.$$

They are equal. This must be true, because if they were not equal, the corresponding heat conduction problem would not have the steady-solution (which we were able to find).

An alternate, mathematical reason for this fact can be given by integrating Laplace's equation over the volume of the annulus. This would yield the fact that the total integral of $\hat{\mathbf{n}} \cdot \nabla u$ over the boundary of the annulus must be zero. Finally, notice that $Q(r = 2)$ is exactly that integral for the part of the boundary at $r = 2$, whereas $Q(r = 1)$ is the negative of the integral corresponding to the boundary at $r = 1$. Thus this fact implies that $Q(1) = Q(2)$.

3 Fourier series (10 points).

(a) Find the Fourier sine series of

$$f(x) = \mathrm{e}^x\,,$$

on the interval $0 \leq x \leq L$.

(b) Find the Fourier cosine series of $f(x)$ on the same interval.

Hint: consider the real and imaginary parts of the integral $\int_0^L \mathrm{e}^{x+in\pi x/L}\,\mathrm{d}x$.

Solution.

(a) The Fourier sine series on $0 < x < L$ is the representation

$$f(x) = \sum_{n=1}^{\infty} a_n \sin\left(\frac{n\pi x}{L}\right)\,,$$

where the a_n are given by

$$a_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)\,\mathrm{d}x\,.$$

With $f(x) = e^x$, the integral can be written

$$a_n = \Im\left[\frac{2}{L}\int_0^L \mathrm{e}^{x(1+\mathrm{i}n\pi/L)}\,\mathrm{d}x\right],$$

where $\Im[z]$ is the imaginary part of z. We find

$$\frac{2}{L}\int_0^L \mathrm{e}^{x(1+\mathrm{i}n\pi/L)}\,\mathrm{d}x = \frac{2}{L+\mathrm{i}n\pi}\mathrm{e}^{x(1+\mathrm{i}n\pi/L)}\Big|_0^L, \tag{1}$$

$$= \frac{2(L-\mathrm{i}n\pi)}{L^2+(n\pi)^2}\left[(-1)^n\mathrm{e}^L - 1\right]. \tag{2}$$

Therefore,

$$a_n = \Im\left[\frac{2}{L}\int_0^L \mathrm{e}^{x(1+\mathrm{i}n\pi/L)}\,\mathrm{d}x\right] = -\frac{2n\pi}{L^2+(n\pi)^2}\left[(-1)^n\mathrm{e}^L - 1\right].$$

(b) The Fourier cosine series on $0 < x < L$ is the representation

$$f(x) = \sum_{n=0}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right),$$

where, the b_n are given by

$$b_n = \frac{2}{L}\int_0^L f(x)\cos\left(\frac{n\pi x}{L}\right)\,\mathrm{d}x.$$

for $n > 1$, while b_0 is

$$b_0 = \frac{1}{L}\int_0^L f(x)\,\mathrm{d}x.$$

Similar to the sine series above, with $f(x) = \mathrm{e}^x$ the integral can be written

$$b_n = \Re\left[\frac{2}{L}\int_0^L \mathrm{e}^{x(1+\mathrm{i}n\pi/L)}\,\mathrm{d}x\right],$$

where $\Re[z]$ denotes the real part of z. With the result we calculated above we find immediately that

$$b_n = \Re\left[\frac{2}{L}\int_0^L \mathrm{e}^{x(1+\mathrm{i}n\pi/L)}\,\mathrm{d}x\right] = \frac{2L}{L^2+(n\pi)^2}\left[(-1)^n\mathrm{e}^L - 1\right].$$

For b_0, we just take $n = 0$ and divide by 2 to get

$$b_0 = \frac{1}{L}\left[\mathrm{e}^L - 1\right].$$

4 Green's functions (10 points). Consider steady-state heat conduction in a rod with an arbitrary source $Q(x)$, non-uniform cross-section, and length L. At $x = 0$ the rod is insulated, so there is zero heat flux out of the rod at $x = 0$. At $x = L$ the rod is held at constant temperature.

(a) Explain why the governing equation is

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(A\frac{\mathrm{d}u}{\mathrm{d}x}\right) + Q = 0.$$

(b) The rod is insulated at $x = 0$ such that the heat flux is zero there. What is the corresponding mathematical boundary condition on u at $x = 0$?

(c) Explain why we are mathematically justified in taking $u = 0$ at the end $x = L$, rather than requiring the actual value of the temperature there.

(d) Find the Green's function for the above inhomogeneous equation with $A = (2L - x)^{-1}$.

(e) Express the solution a sum of two integrals. [*Hint: be very clear about which variable is the variable of integration as well as its range in each integral.*]

Solution.

1. It's the heat equation for variable area.

2. Insulation at $x = 0$ implies
$$\frac{\partial u}{\partial x} = 0$$
there. Constant temperature at $x = L$ implies $u = 0$ there . We can use $u = 0$ because adding or subtracting a constant from u leaves the governing equation unchanged.

3. The Green's function is defined by
$$\frac{\mathrm{d}}{\mathrm{d}x}\left(A\frac{\mathrm{d}G}{\mathrm{d}x}\right) = \delta(x - \xi)\,.$$
We have $A = (2L - x)^{-1}$, integrating once yields
$$\frac{\mathrm{d}G}{\mathrm{d}x} = c(2L - x)\,.$$
Integrating again yields
$$G = c\left(2Lx - \tfrac{1}{2}x^2\right) + b\,.$$
Notice that the left side of ξ, G is constant because of the condition $\mathrm{d}G/\mathrm{d}x = 0$ at $x = 0$. On the right side of ξ we find $b = -3cL^2/2$. Defining $a = c/2$, we can write G as
$$G(x,\xi) = \begin{cases} b & \text{for} \quad x < \xi\,, \\ a[x\,(4L - x) - 3L^2] & \text{for} \quad x > \xi\,. \end{cases}$$

We have that $G(\xi^+) = G(\xi^-)$, which implies $b = a[\xi(4L-\xi) - 3L^2]$, and

$$G(x,\xi) = \begin{cases} a[\xi(4L-\xi) - 3L^2] & \text{for} \quad x < \xi\,, \\ a[\,(4Lx - x^2) - 3L^2] & \text{for} \quad x > \xi\,. \end{cases}$$

and G':

$$G'(x,\xi) = \begin{cases} 0 & \text{for} \quad x < \xi\,, \\ 2a\,(2L - x) & \text{for} \quad x > \xi\,. \end{cases}$$

The jump condition is

$$G'(\xi^+) - G'(\xi^-) = (2L - \xi)\,,$$

which implies $a = \frac{1}{2}$. Good. The Green's function is then

$$G(x,\xi) = \begin{cases} \frac{1}{2}[\xi(4L-\xi) - 3L^2] & \text{for} \quad x < \xi\,, \\ \frac{1}{2}[\,(4Lx - x^2) - 3L^2] & \text{for} \quad x > \xi\,. \end{cases}$$

4. The general solution is

$$y(x) = -\int_0^L Q(\xi) G(x,\xi)\,\mathrm{d}\xi\,, \tag{3}$$

$$= -\tfrac{1}{2}[x(4L-x) - 3L^2]\int_0^x Q(\xi)\,\mathrm{d}\xi - \tfrac{1}{2}\int_x^L [\xi(4L-\xi) - 3L^2] Q(\xi)\,\mathrm{d}\xi\,. \tag{4}$$

5 Fourier transform: wave equation with damping (10 points). Consider the equation

$$\frac{\partial^2 u}{\partial t^2} + 2\gamma\frac{\partial u}{\partial t} = c^2\frac{\partial^2 u}{\partial x^2}$$

with initial conditions

$$u(x,0) = f(x) \qquad \text{and} \qquad \frac{\partial u}{\partial t}(x,0) = 0\,.$$

(a) What are the units of γ? What sign do you expect γ to have for well-behaved solutions? [*Hint: You can ignore the right-hand side to answer this.*]

(b) Obtain the equation for the Fourier transform $U(\omega,t)$ of $u(x,t)$. [*Hint: you will find that both solutions for U have the form $U \sim \mathrm{e}^{\alpha t}$. To save yourself some ink, write α as $\alpha = -\gamma \pm \mathrm{i}\sigma$, and give the form of σ in terms of c, γ, and ω.*]

(c) Let $f(x) = \delta(x)$ and find the physical space solution $u(x,t)$. You may leave your answer as an integral.

(d) Define the physical space solution found in (c) as $G(x,t)$. Write the general solution for $u(x,t)$ as a convolution integral involving $G(x,t)$ and an arbitrary initial condition $f(x)$.

Solution.

1. The units of γ are
$$\gamma = \left[\frac{\frac{\partial^2 u}{\partial t^2}}{\frac{\partial u}{\partial t}}\right] = \frac{1}{\text{time}}\,.$$
γ should be greater than zero. This produces solutions with exponential decay, rather than exponential growth.

2. The Fourier transform of the wave equation is
$$\frac{\partial^2 U}{\partial t^2} + 2\gamma\frac{\partial U}{\partial t} + c^2\omega^2 U = 0\,.$$
The solutions are exponentials of the form $U = Ce^{\alpha t}$. The characteristic equation for α is
$$\alpha^2 + 2\gamma\alpha + c^2\omega^2 = 0\,.$$
The quadratic equation then implies that
$$\alpha = -\gamma \pm \sqrt{\gamma^2 - c^2\omega^2}\,.$$
Define $\sigma = \sqrt{c^2\omega^2 - \gamma^2}$, so that $\alpha = -\gamma \pm \mathrm{i}\sigma$. The general solution is
$$U = \mathrm{e}^{-\gamma t}\left(A\mathrm{e}^{\mathrm{i}\sigma t} + B\mathrm{e}^{-\mathrm{i}\sigma t}\right)\,.$$
The initial condition $\partial U/\partial t = 0$ requires
$$0 = \frac{\partial U}{\partial t}(t=0) = (-\gamma + \mathrm{i}\sigma)\,A + (-\gamma - \mathrm{i}\sigma)\,B\,,$$
which implies
$$B = A\frac{-\gamma + \mathrm{i}\sigma}{\gamma + \mathrm{i}\sigma} = -A\frac{(\gamma - \mathrm{i}\sigma)^2}{\gamma^2 + \sigma^2}\,.$$
The solution is therefore
$$U = A\mathrm{e}^{-\gamma t}\left(\mathrm{e}^{\mathrm{i}\sigma t} - \frac{(\gamma - \mathrm{i}\sigma)^2}{\gamma^2 + \sigma^2}\mathrm{e}^{-\mathrm{i}\sigma t}\right)\,.$$
The condition $U(\omega, 0) = F(\omega)$ then implies that
$$F(\omega) = A\left(1 - \frac{(\gamma - \mathrm{i}\sigma)^2}{\gamma^2 + \sigma^2}\right)\,,$$
or in other words,
$$A = F(\omega)\frac{\gamma^2 + \sigma^2}{2\sigma(\sigma + \mathrm{i}\gamma)}\,.$$
The solution is therefore
$$U(\omega, t) = \frac{\gamma^2 + \sigma^2}{2\sigma(\sigma + \mathrm{i}\gamma)}F(\omega)\mathrm{e}^{-\gamma t}\left(\mathrm{e}^{\mathrm{i}\sigma t} - \frac{(\gamma - \mathrm{i}\sigma)^2}{\gamma^2 + \sigma^2}\mathrm{e}^{-\mathrm{i}\sigma t}\right)$$

3. When $f(x) = \delta(x)$, we have $F(\omega) = (2\pi)^{-1}$, and

$$G(x,t) = (2\pi)^{-1} \mathrm{e}^{-\gamma t} \int_{-\infty}^{\infty} \frac{\gamma^2 + \sigma^2}{2\sigma(\sigma + \mathrm{i}\gamma)} \left(\mathrm{e}^{-\mathrm{i}(\omega x - \sigma t)} - \frac{(\gamma - \mathrm{i}\sigma)^2}{\gamma^2 + \sigma^2} \mathrm{e}^{\mathrm{i}(\omega x + \sigma t)} \right) \mathrm{d}\omega \,,$$

where

$$\sigma \stackrel{\text{def}}{=} \sqrt{(c\omega)^2 - \gamma^2} \,.$$

6 Using D'Alembert's solution (10 points). Consider the wave equation in an infinite domain:

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} \,, \qquad \text{where} \qquad -\infty < x < \infty \,,$$

and the initial conditions on $u(x,t)$ are

$$u(x,0) = 0 \qquad \text{and} \qquad \frac{\partial u}{\partial t}(x,0) = g(x) \,.$$

1. Write down the general solution $u(x,t)$ for arbitrary $g(x)$.
2. Compute and draw the solution at *two* later times when

$$g(x) = x\,\mathrm{e}^{-x^2/2} \,.$$

[*Hint: because the function* $x\,\mathrm{e}^{-x^2/2}$ *is an exact derivative, it can be integrated in closed form.*]

Solution.

1. The general solution is

$$u(x,t) = \int_{x-ct}^{x+ct} g(\xi)\,\mathrm{d}\xi \,.$$

2. When $g(x) = x\,\mathrm{e}^{-x^2/2}$, the general solution is

$$u(x,t) = \int_{x-ct}^{x+ct} \xi \mathrm{e}^{-\xi^2/2}\,\mathrm{d}\xi = -\mathrm{e}^{-\xi^2/2}\Big|_{x-ct}^{x+ct} = \mathrm{e}^{-\frac{1}{2}(x-ct)^2} - \mathrm{e}^{-\frac{1}{2}(x+ct)^2} \,.$$

The solution looks consists of two Gaussians, one positive and propagating in the positive x-direction, and the other negative and propagating in the negative x-direction.