Homework I

Due April 8, 2015.

1 Trigonometry.

(a) Using the two identities

$$\sin(a \pm b) = \sin(a)\cos(b) \pm \cos(a)\sin(b),$$
$$\cos(a \pm b) = \cos(a)\cos(b) \mp \sin(a)\sin(b),$$

show that

$$\sin^2(x) = \tfrac{1}{2}\Big[1 - \cos(2x)\Big], \tag{1}$$

and

$$\cos^2(x) = \tfrac{1}{2}\Big[1 + \cos(2x)\Big]. \tag{2}$$

(b) Provide an alternative proof of the identities (1) and (2) in part (a) using the fact that

$$\cos(2x) + \mathrm{i}\sin(2x) = \mathrm{e}^{2\mathrm{i}x} = \Big[\cos(x) + \mathrm{i}\sin(x)\Big]^2.$$

(c) Showing all intermediate steps, compute the following integrals:

i. $$I_1 = \int_0^{2\pi} \sin^2(x)\,\mathrm{d}x,$$

ii. $$I_2 = \int_0^{2\pi} \cos(x)\cos(3x)\,\mathrm{d}x.$$

2 Ordinary differential equations I. Find the general solution $y(x)$ to the following ordinary differential equations:

a) $y'' + 25y = 0,$

b) $y'' - 25y = 0,$

c) $y'' + 2y' + y = 0,$

d) $y'' + 2y' + 6y = 0.$

3 Ordinary differential equations II.

(a) Find the solution $y(x)$ to the ordinary differential equation

$$y' - 4y = 0, \tag{3}$$

subject to the condition that

$$y(0) = 1.$$

(b) Solve the problem in part 1 using separation of variables. In other words, observe that $y' = \mathrm{d}y/\,\mathrm{d}x$ implies that equation (3) can be written

$$\frac{\mathrm{d}y}{y} = 4\,\mathrm{d}x. \tag{4}$$

Next, integrate equation (4) to find

$$\int \frac{\mathrm{d}y}{y} = \int 4\,\mathrm{d}x,$$

and use the condition $y(0) = 1$ to determine the constant of integration. Show all steps.

4 Steady heat conduction I. Consider a constant-area steel wire extending from $x = 0$ to $x = 1$ in non-dimensional units and heated at a constant rate S. The non-dimensional heat conduction equation for the temperature $T(x,t)$ of the wire is

$$\frac{\partial T}{\partial t} = k\frac{\partial^2 T}{\partial x^2} + S.$$

Now imagine that the temperature of the wire at $x = 0$ is fixed at $T = 0$, and that the heat flux out of the wire at $x = 1$ is somehow fixed at $\partial T/\partial x = f$. Whatever the initial temperature distribution is in the wire, it will eventually approach a steady-state where $\partial T/\partial t = 0$. Find the steady temperature distribution $T(x)$ in terms of f and S.

5 Steady heat conduction II. Consider a metal rod 1 meter long. We define x so that the ends of the rod are at $x = 0$ and $x = 1$ m. This particular rod has a curious shape in that its area $A(x)$ varies according to

$$A(x) = \frac{A_0}{1 + x^2/5},$$

where $A_0 = 1\ \mathrm{m}^2$ is the area of the rod at $x = 0$. The heat conductivity of the rod is $k = 100\ \mathrm{W/m\,^\circ C}$. Assume the temperature of the rod depends only on x; in other words, the temperature of the rod is governed by the one-dimensional, variable-area heat conduction equation

$$\frac{\partial T}{\partial t} = \frac{\partial}{\partial x}\left(kA\frac{\partial T}{\partial x}\right).$$

Note that the area, $A(x)$, is a function of x. The temperature of the rod at either end is fixed: at $x = 0$ the rod temperature is $T = 18\,^\circ\mathrm{C}$, and at $x = 1$ m the rod temperature is $T = 22\,^\circ\mathrm{C}$. Answer the following:

1. What is the steady-state temperature distribution $T(x)$ in the rod in degrees Celsius?
2. Why may we use degrees Celsius in this problem, rather than degrees Kelvin, which were the units in which equation () was originally derived?