Solutions I

1 Trigonometry.

(a) If we take $a = b = x$, the identities for $\cos(a+b)$ and $\cos(a-b)$ become

$$\cos(2x) = \cos^2(x) - \sin^2(x)\,, \tag{1}$$
$$1 = \cos^2(x) + \sin^2(x)\,. \tag{2}$$

We can use the identities in (1) and (2) to prove the two identities in part (a). Subtracting equation (1) from (2) yields

$$1 - \cos(2x) = \cos^2(x) + \sin^2(x) - \Big(\cos^2(x) - \sin^2(x)\Big) = 2\sin^2(x)\,. \tag{3}$$

Division by 2 then yields identity (a) for $\sin^2(x)$,

$$\sin^2(x) = \tfrac{1}{2}\Big[1 - \cos(2x)\Big]\,.$$

The second identity for $\cos^2(x)$ can be obtained in a similar manner, except that we add (1) and (2). This yields

$$\cos(2x) + 1 = 2\cos^2(x)\,, \tag{4}$$

and division by 2 yields identity (b) for $\cos^2(x)$.

(b) If we expand the squared quantity, we find

$$e^{2ix} = \Big[\cos(x) + i\sin(x)\Big]^2 = \cos^2(x) + 2i\cos(x)\sin(x) - \sin^2(x)\,. \tag{5}$$

But we also have that

$$e^{2ix} = \cos(2x) + i\sin(2x)\,. \tag{6}$$

The real and imaginary parts of (5) and (6) must be equal. Therefore

$$\cos^2(x) - \sin^2(x) = \cos(2x) \quad \text{and} \quad 2\cos(x)\sin(x) = \sin(2x)\,. \tag{7}$$

We can use the first identity, along with the identity $\cos^2(x) + \sin^2(x) = 1$, to prove identities (a) and (b) as previously.

(c) For i., use identity (a) to write

$$I_1 = \int_0^{2\pi} \sin^2(x)\,\mathrm{d}x = \int_0^{2\pi} \left(\tfrac{1}{2} - \tfrac{1}{2}\cos(2x)\right)\,\mathrm{d}x = \tfrac{1}{2}x\,\Big|_0^{2\pi} - \tfrac{1}{4}\sin(2x)\,\Big|_0^{2\pi} = \pi\,. \quad (8)$$

Therefore $I_1 = \pi$.

For ii., we require the identity

$$\cos(a)\cos(b) = \tfrac{1}{2}\left[\cos\left([a-b]x\right) + \cos\left([a+b]x\right)\right],$$

which implies that

$$\cos(x)\cos(3x) = \tfrac{1}{2}\Big(\cos(2x) + \cos(4x)\Big)\,.$$

We can use this to write I_2 as

$$\begin{aligned} I_2 &= \int_0^{2\pi} \cos(x)\cos(3x)\,\mathrm{d}x = \int_0^{2\pi} \tfrac{1}{2}\cos(2x) + \tfrac{1}{2}\cos(4x)\,\mathrm{d}x \\ &= \tfrac{1}{4}\sin(2x)\,\Big|_0^{2\pi} + \tfrac{1}{8}\sin(4x)\,\Big|_0^{2\pi} = 0\,. \end{aligned} \quad (9)$$

Therefore $I_2 = 0$.

2 a) Try the form $y(x) = \mathrm{e}^{kx}$. Plugging this in yields

$$\mathrm{e}^{kx}\left(k^2 + 25\right) = 0\,,$$

The “ associated” or “characteristic” equation corresponds to setting the parenthetical quantity to zero (the other option $\mathrm{e}^{kx} = 0$ corresponds to choosing the “trivial” solution $y = 0$). The associated equation is then

$$k^2 + 25 = 0\,.$$

The two solutions to this equation are $k_1 = 5\mathrm{i}$ and $k_2 = -5\mathrm{i}$, where $\mathrm{i} = \sqrt{-1}$ is the imaginary unit. The two possible solutions for $y(x)$ are then $y_1(x) = A\mathrm{e}^{5\mathrm{i}x}$ and $y_2 = B\mathrm{e}^{-5\mathrm{i}x}$, where A and B are arbitrary constants. Because the governing equation is linear, the general solution is a linear combination of these two solutions, so

$$y(x) = A\mathrm{e}^{5\mathrm{i}x} + B\mathrm{e}^{-5\mathrm{i}x}\,.$$

Using Euler’s identity $\mathrm{e}^{\mathrm{i}\theta} = \cos\theta + \mathrm{i}\sin\theta$ we can write this in the alternative (and possibly more convenient if the boundary conditions are given as real quantities) form

$$y(x) = C\cos(x) + D\sin(x)\,,$$

where C and D are given by $C = A + B$ and $D = \mathrm{i}(A - B)$.

b) The associated equation is

$$k^2 - 25 = 0\,,$$

and the two solutions for k are $k = 5$ and $k = -5$. The general solution for $y(x)$ is therefore

$$y(x) = A\mathrm{e}^{5x} + B\mathrm{e}^{-5x}\,.$$

c) The associated equation is

$$0 = k^2 + 2k + 1 = (k+1)^2\,. \tag{10}$$

This equation has a "repeated root" at $k = -1$. (Because the equation is a second-order polynomial, it has two solutions, which are both $k = -1$.) When the associated equation has one repeated root k, the differential equation has one solution given by $y_1 = A\mathrm{e}^{kx}$, and another given by $y_2 = Bx\mathrm{e}^{kx}$. This can be verified by substituting $y_1(x)$ and $y_2(x)$ into the original differential equation and observing that both do satisfy the differential equation. The general solution is therefore

$$y(x) = A\mathrm{e}^{-x} + Bx\mathrm{e}^{-x}\,.$$

d) The associated equation is

$$0 = k^2 + 2k + 6\,.$$

The two solutions to this equation can be found using the quadratic formula:

$$k_1 = -1 + \sqrt{5}\mathrm{i} \qquad \text{and} \qquad k_2 = -1 - \sqrt{5}\mathrm{i}\,.$$

The general solution for $y(x)$ is therefore

$$y(x) = A\mathrm{e}^{(-1+\sqrt{5}x} + B\mathrm{e}^{(-1-\sqrt{5})x}\,.$$

Using Euler's identity we can write this in the potentially more convenient and intuitive form

$$y(x) = \mathrm{e}^{-x}\Big[C\cos(\sqrt{5}x) + D\sin(\sqrt{5}x)\Big]\,.$$

3 Ordinary differential equations II.

a) The ordinary differential equation has the associated equation

$$k - 4 = 0\,,$$

which gives $k = 4$ and the general solution $y(x) = A\mathrm{e}^{4x}$. We apply the condition $y(0) = 1$ and obtain $A = 1$. The solution is hence

$$y(x) = \mathrm{e}^{4x}\,.$$

It can be verified that this satisfies the differential equation and the condition $y(0) = 1$.

b) We separate variables by writing the differential equation as

$$\frac{\mathrm{d}y}{\mathrm{d}x} = 4y\,,$$

so dividing by y and multiplying by $\mathrm{d}x$ yields

$$\frac{\mathrm{d}y}{y} = 4\,\mathrm{d}x\,.$$

We can then integrate both sides. We have the indefinite integrals

$$\int \frac{\mathrm{d}y}{y} = \ln y + A\,, \quad \text{and} \quad \int 4\,\mathrm{d}x = 4x + B\,.$$

We thus find that

$$\ln y = 4x + C\,,$$

where A is the single undetermined constant in the problem. We then take the exponential of this equation. Since $\mathrm{e}^{\ln y} = y$, we find

$$y = \mathrm{e}^{4x+C} = D\mathrm{e}^{4x}\,,$$

where we have introduced a new undetermined constant $D = \mathrm{e}^C$ for convenience. As in part a), applying the condition $y(0) = 1$ yields $D = 1$ and the solution

$$y(x) = \mathrm{e}^{4x}\,.$$

4 In steady state, the temperature of the wire is governed by

$$k\frac{\mathrm{d}^2T}{\mathrm{d}x^2} + S = 0\,.$$

Since k and S are constants, we can integrate once to obtain

$$\frac{\mathrm{d}T}{\mathrm{d}x} = \left(\frac{S}{k}\right)x + A\,,$$

where A is a constant. We can either integrate again, or apply the boundary condition (which is a simpler option here). Applying the condition $\partial T/\partial x = f$ at $x = 1$ implies that

$$f = \frac{S}{k} + A\,,$$

and so $A = f - S/k$. We have then

$$\frac{\mathrm{d}T}{\mathrm{d}x} = \frac{S}{k}\,(x-1) + f\,.$$

Integrating again, we find

$$T(x) = \frac{S}{k}\left(\tfrac{1}{2}x^2 - x\right) + fx + B\,.$$

The condition $T(x = 0) = 0$ determines B, and we find $B = 0$. The final solution is therefore

$$T(x) = \frac{S}{2k}x^2 + (f - S/k)\,x\,.$$

5 Before we solve this problem, we note an error in the problem statement. The variable-area heat condition equation is actually

$$A\frac{\partial T}{\partial t} = \frac{\partial}{\partial x}\left(kA\frac{\partial T}{\partial x}\right)\,.$$

This error does not affect the solutions. Another note concerns about dimensions. The form given for $A(x)$ is dimensional; in particular the dimensions of $A(x)$ are determined by A_0, which, naturally, has dimensions of area. This implies that the denominator $1 + x^2/5$ is dimensionless. Thus, because x has dimensions of meters, we must have that "x^2" has dimensions meters^2, and that "5" is properly interpreted as also having dimensions of meters^2. This unfortunate confusion would have been avoided if the problem statement had been written

$$A(x) = \frac{A_0}{1 + x^2/A_1}\,,$$

where $A_0 = 1\,\mathrm{m}^2$ and $A_1 = 5\,\mathrm{m}^2$. In any case, on to the answers...

1. In steady-state, we have $\partial T/\partial t = 0$, and the variable-area heat conduction equation becomes

   $$k\frac{\mathrm{d}}{\mathrm{d}x}\left(A\frac{\mathrm{d}T}{\mathrm{d}x}\right) = 0\,,$$

   where we have moved k outside of the derivative because it is constant. We first divide both sides by k to eliminate it. We can then integrate once to find

   $$A\frac{\mathrm{d}T}{\mathrm{d}x} = C\,,$$

   where C is a constant. Our equation for $\mathrm{d}T/\mathrm{d}x$ is therefore

   $$\frac{\mathrm{d}T}{\mathrm{d}x} = CA(x) = C\left(1 + \tfrac{1}{5}x^2\right)\,.$$

   Integrating again to find T, we find

   $$T = C\left(x + \tfrac{1}{15}x^3\right) + D\,,$$

   where D is another constant. The condition $T(0) = 18\,^\circ C$ implies that $D = 18\,^\circ C$. The condition $T(1) = 22\,^\circ C$ implies that

   $$22\,^\circ C = C\frac{16}{15}\,\mathrm{m} + 18\,^\circ C\,,$$

and therefore

$$C = 4\,^\circ\mathrm{C} \times \frac{15}{16}\frac{1}{\mathrm{m}} = \frac{15}{4}\,^\circ\mathrm{C}/\mathrm{m}\,.$$

We thus have for $T(x)$,

$$T(x) = \tfrac{15}{4}\left(x + \tfrac{1}{15}x^3\right) + 18\,,$$

where x is in meters and T is in $^\circ C$.

2. In the heat conduction equation, we are not concerned with the absolute value of the temperature, but rather its rate of change: its x-gradient, $\partial T/\partial x$ (and $\partial^2 T/\partial x^2$), and its time-rate of change $\partial T/\partial t$. Since Kelvin and Celsius differ only by a constant, the heat-conduction equation is identical whether T is expressed in Kelvin of Celsius. Another way to see this is to substitute

$$T(x,t) = T_K(x,t) + 273.15\,,$$

into the heat equation and derive an equation for $T_K(x,t)$, or temperature in Kelvin. Compare this to the equation for $T(x,t)$: they are the same.