Homework 2

Due April 24, 2015.

1 Separation of Variables: mixed boundary conditions. Consider the heat equation,

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2},$$

with boundary conditions

$$k\frac{\partial u}{\partial x}(x=0,t) = 0, \qquad \text{and} \qquad u(x=L,t) = 0,$$

and the linear initial condition

$$u(x,t=0) = a(x-L).$$

(a) What is the steady-state solution when $\partial u/\partial t = 0$?

(b) Use separation of variables to solve the initial value problem for $u(x,t)$.

2 Separation of Variables: A pretty bad model for combustion. Consider the same heat equation as above, but with source cu (proportional to u), where c is a constant. This heat-dependent source term might (poorly) model a source of heat arising from, for example, a chemical reaction like combustion. The heat equation becomes

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2} + cu.$$

Assume insulating boundary conditions such that

$$\frac{\partial u}{\partial x}(x=0,t) = \frac{\partial u}{\partial x}(x=L,t) = 0.$$

(a) Use separation of variables to find the general solution as a series of cosines.

(b) What is the critical value of c for which $u(x,t)$ can increase in time?

3 Time-dependent forcing: The sun heating the ocean. Let's try to model for how the sun heats the ocean surface. We use the boundary conditions

$$u(z=0,t) = u_0 \mathrm{e}^{\mathrm{i}\omega t} \qquad \text{and} \qquad \frac{\partial u}{\partial z}(z \to -\infty, t) \to 0\,,$$

where $z = 0$ is the ocean surface and as $z \to -\infty$ we are descending into the abyssal depths of the ocean. We propose to model the action of the sun as a source term in the heat equation by solving

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial z^2} + Q_0 \mathrm{e}^{z/\lambda} \mathrm{e}^{\mathrm{i}\omega t}\,.$$

(a) Assume $\omega = 0$ and $\partial u/\partial t = 0$. Find the steady-state solution to the problem.

(b) Now solve the problem with $\omega \neq 0$. Propose $u(z,t) = w(z)\mathrm{e}^{\mathrm{i}\omega t}$, then derive an equation for $w(z)$. Solve this equation.

(c) Write down the real part of your solution for $w(z)$. This is the solution you would find if you replaced "$\mathrm{e}^{\mathrm{i}\omega t}$" in the source term and boundary condition with $\cos(\omega t)$.

3 Laplace's equation in a square. Consider Laplace's equation in Cartesian coordinates in (x, y),

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0\,,$$

with the boundary conditions

$$u(x=0, y) = -1\,, \qquad u(x, y=0) = 0\,,$$
$$u(x=L, y) = 1\,, \qquad u(x, y=L) = 0\,.$$

A sketch is given below.

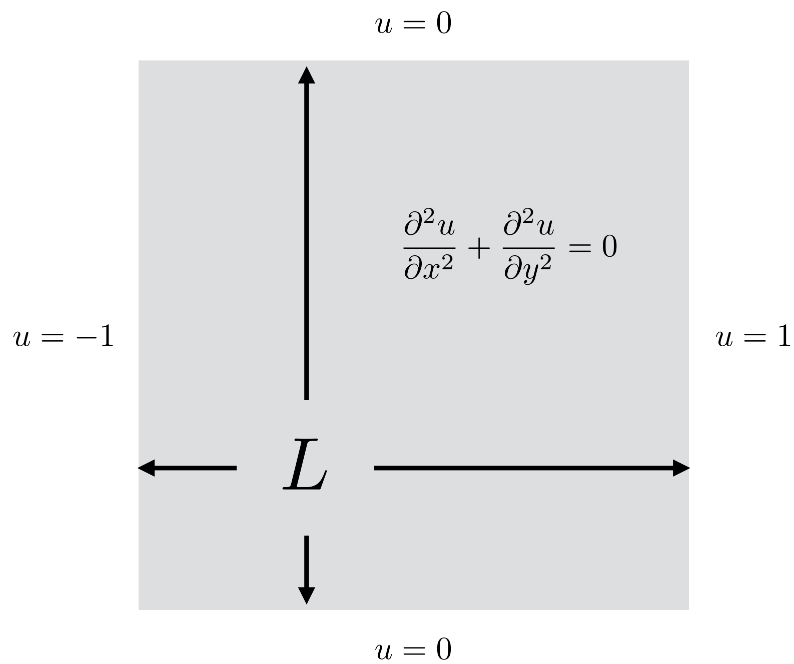

Figure 1: "Sketch" for problem 4.

(a) Use the principle of superposition and separation of variables to find $u(x, y)$ which satisfies the governing equation and all boundary conditions.

(b) What is the solution for $u(x, y)$ when the boundary conditions at $y = 0$ and $y = L$ are both changed to

$$\frac{\partial u}{\partial y} = 0\,?$$

Finding the solution should not require more than a line or two of calculation. *Hint: will the solution depend on y?*

4 Laplace's equation outside a disk. Consider Laplace's equation outside the disk with radius a. The domain thus extends from $r = a$ to ∞. Laplace's equation in polar coordinates is

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0\,.$$

The boundary condition at $r = a$ is

$$\frac{\partial u}{\partial r}(r = a, \theta) = 1 + 2\sin\theta\,.$$

The boundary condition as $r \to +\infty$ is

$$\nabla u(r \to \infty, \theta) \to 0\,.$$

A sketch is below.

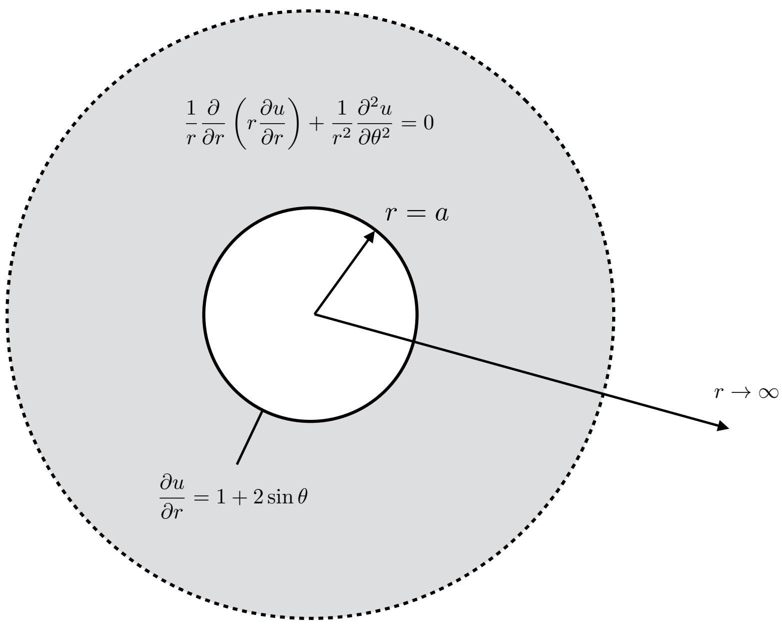

Figure 2: "Sketch" for problem 5.

(a) Solve for $u(r, \theta)$. *Hint: your solution will contain an undeterminable constant. This is because there are two solutions with no θ-dependence.*