Homework 2 Solutions

Due April 24, 2015.

1 Separation of Variables: mixed boundary conditions. Consider the heat equation,

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2},$$

with boundary conditions

$$k\frac{\partial u}{\partial x}(x = 0, t) = 0, \qquad \text{and} \qquad u(x = L, t) = 0,$$

and the linear initial condition

$$u(x, t = 0) = a(x - L).$$

(a) What is the steady-state solution when $\partial u/\partial t = 0$?

(b) Use separation of variables to solve the initial value problem for $u(x, t)$.

Solution.

(a) The steady-state solution is $u(x, t) = 0$.

(b) We use separation of variables by proposing $u = f(x)g(t)$. By substituting this into the governing equation and dividing by kfg, we find

$$\frac{g'}{kg} = \frac{f''}{f} = -\lambda,$$

where we have set the two sides of the equation equal to a separation constant $-\lambda$. We thus obtain two ODE's for $f(x)$ and $g(t)$. The equation for $g(t)$ is

$$g' + k\lambda g = 0,$$

which has the solution

$$g = Ce^{-k\lambda t}.$$

The equation for $f(x)$ is

$$f'' + \lambda f = 0,$$

and the solution is

$$f = A\sin\left(\sqrt{\lambda}x\right) + B\cos\left(\sqrt{\lambda}x\right) .$$

The boundary conditions on $f(x)$ follow from the boundary conditions on $u(x,t)$, and are $f'(0) = 0$ and $f(L) = 0$. The condition $f'(0) = 0$ implies that $A = 0$. The condition at $x = L$ implies

$$0 = B\cos\left(\sqrt{\lambda}L\right) .$$

The only non-trivial solutions are thus found when $\cos(\sqrt{\lambda}L) = 0$. Being well-acquainted with the cosine function, we know it will have zeros when $\sqrt{\lambda}L$ is $\pi/2$, $3\pi/2$, etc. Another way to say this is $\sqrt{\lambda}L = (n-1/2)\pi$ for n a positive integer. Therefore we find that the spatial modes have the form

$$f = B\cos\left[\left(n-\tfrac{1}{2}\right)\frac{\pi x}{L}\right] .$$

The general solution for u is a sum of all modes, and is thus

$$u(x,t) = \sum_{n=1}^{\infty} B_n \cos\left[\left(n-\tfrac{1}{2}\right)\frac{\pi x}{L}\right] \exp\left\{-\left(n-\tfrac{1}{2}\right)^2 \frac{\pi^2}{L^2}kt\right\} .$$

We use the initial condition to determine the coefficients B_n. At $t = 0$ we have

$$u(x,0) = a(x-L) = \sum_{n=1}^{\infty} B_n \cos\left[\left(n-\tfrac{1}{2}\right)\frac{\pi x}{L}\right] .$$

To find the B_n's, we multiply this equation by $\cos((m-1/2)\pi x/L)$ and integrate from 0 to L. This yields

$$B_n = \frac{2}{L}\int_0^L a(x-L)\cos\left[\left(n-\tfrac{1}{2}\right)\frac{\pi x}{L}\right]\, \mathrm{d}x .$$

We can evaluate this integral fairly easily with integration by parts. We find

$$B_n = -\frac{2aL}{\left(n-\frac{1}{2}\right)^2\pi^2} = -\frac{8aL}{(2n-1)^2\pi^2} .$$

The solution is therefore

$$u(x,t) = -\sum_{n=1}^{\infty} \frac{8aL}{(1-2n)^2\pi}\cos\left[\left(n-\tfrac{1}{2}\right)\frac{\pi x}{L}\right] \exp\left\{-\left(n-\tfrac{1}{2}\right)^2 \frac{\pi^2}{L^2}kt\right\} .$$

A plot of the evolution of $u(x,t)$ is shown in Figure 1.

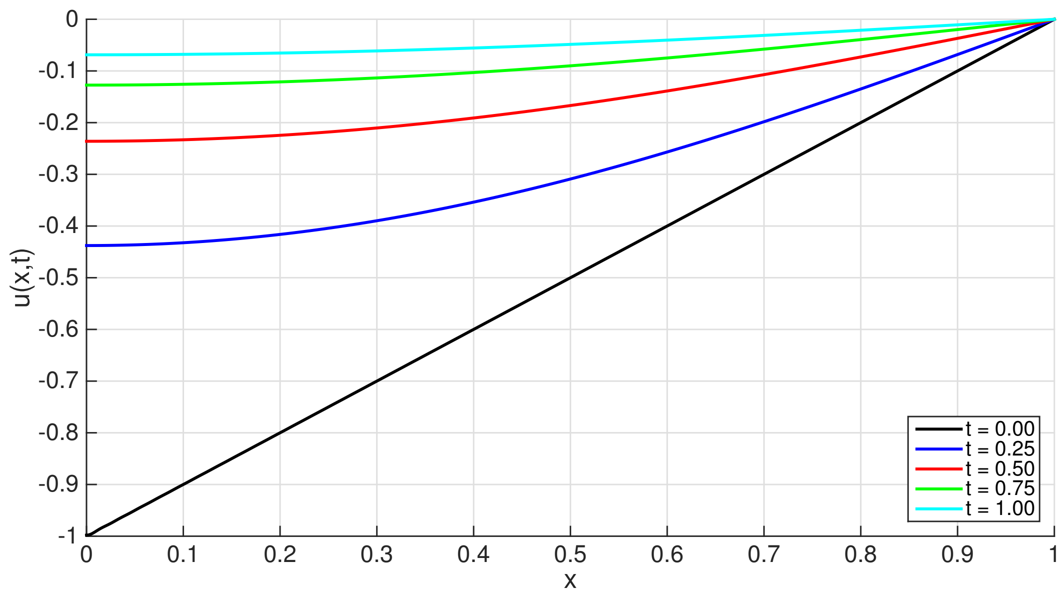

Figure 1: The solution $u(x,t)$ to 1(b) at various times t.

2 Separation of Variables: A pretty bad model for combustion. Consider the same heat equation as above, but with source cu (proportional to u), where c is a constant. This heat-dependent source term might (poorly) model a source of heat arising from, for example, a chemical reaction like combustion. The heat equation becomes

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2} + cu\,.$$

Assume insulating boundary conditions such that

$$\frac{\partial u}{\partial x}(x=0,t) = \frac{\partial u}{\partial x}(x=L,t) = 0\,.$$

(a) Use separation of variables to find the general solution as a series of cosines.

(b) What is the critical value of c for which $u(x,t)$ can increase in time?

Solution.

(a) To use separation of variables, we propose u of the form $u(x,t) = f(x)g(t)$. Substituting this into the governing equation yields

$$fg' = kgf'' + cfg\,.$$

Next, we multiply by $1/kfg$, and move the "c" term over to the left side. This gives

$$\frac{g'}{kg} - \frac{c}{k} = \frac{f''}{f}\,.$$

Per the usual separation of variables argument, because either side is dependent either on x or t but not both, we can only conclude that they are equal to a constant, which we call $-\lambda$. Note that we could have put the constant term c/k on either side; it only seems that including it with the g-equation is simpler.

The f-equation is

$$f'' + \lambda f = 0\,.$$

We showed in class that, with the boundary condition $f'(0) = f'(L) = 0$, the solutions are cosines of the form

$$f(x) = A\cos\left(\frac{n\pi x}{L}\right)\,,$$

where we have found that $\lambda = (n\pi/L)^2$. It is easy to verify that this satisfies the boundary conditions. The g-equation is

$$g' + (k\lambda - c)g = 0\,,$$

and the solution is

$$g = C\mathrm{e}^{-k\lambda t}\mathrm{e}^{ct}\,.$$

The general solution is therefore

$$u(x,t) = \mathrm{e}^{ct}\sum_{n=1}^{\infty} B_n \cos\left(\frac{n\pi x}{L}\right)\exp\left\{-k\left(\frac{n\pi}{L}\right)^2 t\right\}\,.$$

(b) Because of the e^{ct} term in front, it is possible for $u(x,t)$ to increase in time, as opposed to decreasing like we normally expect. Note that the modes of $u(x,t)$ (corresponding to values of n) decay at different rates – and the first mode ($n = 1$) decays the slowest. Thus as c increases from 0, it is the first mode which first can grow. The time-dependence of the first mode is

$$\exp\left\{\left(c - \frac{k\pi^2}{L^2}\right)t\right\}\,.$$

Thus, the amplitude of the first mode grows in time when

$$c > \frac{k\pi^2}{L^2}\,.$$

When c is less than $k\pi^2/L^2$, none of the modes can grow and the solution decreases in time. When c equals $k\pi^2/L^2$, the amplitude of the first mode is constant, while the higher modes decay to zero.

3 Time-dependent forcing: The sun heating the ocean. Let's try to model for how the sun heats the ocean surface. We use the boundary conditions

$$u(z=0,t) = u_0\mathrm{e}^{\mathrm{i}\omega t} \qquad \text{and} \qquad \frac{\partial u}{\partial z}(z\to -\infty, t) \to 0\,,$$

where $z = 0$ is the ocean surface and as $z \to -\infty$ we are descending into the abyssal depths of the ocean. We propose to model the action of the sun as a source term in the heat equation by solving

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial z^2} + Q_0 \mathrm{e}^{z/\lambda}\mathrm{e}^{\mathrm{i}\omega t}\,.$$

(a) Assume $\omega = 0$ and $\partial u/\partial t = 0$. Find the steady-state solution to the problem.

(b) Now solve the problem with $\omega \neq 0$. Propose $u(z,t) = w(z)\mathrm{e}^{\mathrm{i}\omega t}$, then derive an equation for $w(z)$. Solve this equation.

(c) Write down the real part of your solution for $w(z)$. This is the solution you would find if you replaced "$\mathrm{e}^{\mathrm{i}\omega t}$" in the source term and boundary condition with $\cos(\omega t)$.

Solution.

(a) When $\omega = 0$ and $\partial u/\partial t = 0$, the governing equation reduces to

$$0 = k\frac{\mathrm{d}^2 u}{\mathrm{d}z^2} + Q_0 \mathrm{e}^{z/\lambda}\,,$$

with the boundary conditions $u(0,t) = u_0$ and $\mathrm{d}u/\mathrm{d}z \to 0$ as $z \to -\infty$. This equation is easily integrated. First we rearrange it to obtain,

$$\frac{\mathrm{d}^2 u}{\mathrm{d}z^2} = -\frac{Q_0}{k}\mathrm{e}^{z/\lambda}\,.$$

One integration yields

$$\frac{\mathrm{d}u}{\mathrm{d}z} = -\frac{\lambda Q_0}{k}\mathrm{e}^{z/\lambda} + A\,,$$

and another yields

$$u = -\frac{\lambda^2 Q_0}{k}\mathrm{e}^{z/\lambda} + Az + B\,.$$

The condition that $\mathrm{d}u/\mathrm{d}z \to 0$ as $z \to -\infty$ means that $A = 0$ (otherwise $\mathrm{d}u/\mathrm{d}z$ equals a constant as $z \to -\infty$). The condition at $z = 0$ implies

$$u_0 = -\frac{\lambda^2 Q_0}{k} + B\,,$$

and from this we deduce that

$$B = \frac{\lambda^2 Q_0}{k} + u_0\,.$$

The steady-state solution is then

$$u(z,t) = u_0 + \frac{\lambda^2 Q_0}{k}\left(1 - \mathrm{e}^{z/\lambda}\right)\,.$$

(b) We propose $u(z,t) = w(z)\mathrm{e}^{\mathrm{i}\omega t}$. Substituting this into the governing equation yields

$$w\left(\mathrm{i}\omega \mathrm{e}^{\mathrm{i}\omega t}\right) = k\mathrm{e}^{\mathrm{i}\omega t}w'' + Q_0\mathrm{e}^{z/\lambda}\mathrm{e}^{\mathrm{i}\omega t}\,.$$

Because every term depends on $\mathrm{e}^{\mathrm{i}\omega t}$, we can remove this from the equation. This yields

$$w'' - \frac{\mathrm{i}\omega}{k}w = -\frac{Q_0}{k}\mathrm{e}^{z/\lambda}\,.$$

The boundary conditions on w are $w = u_0$ at $x = 0$ and $w' \to 0$ as $z \to -\infty$. The homogeneous solution to this equation (the part of the solution that satisfies $w'' - \mathrm{i}\omega w/k = 0$) is

$$w_h = A\mathrm{e}^{z\sqrt{\mathrm{i}\omega/k}} + B\mathrm{e}^{-z\sqrt{\mathrm{i}\omega/k}}\,.$$

The particular part of the solution can be found by guessing a solution of the form $w_p = C\mathrm{e}^{z/\lambda}$. Plugging this into the differential equation yields

$$\mathrm{e}^{z/\lambda}C\left(\lambda^{-2} - \mathrm{i}\omega/k\right) = -\frac{Q_0}{k}\mathrm{e}^{z/\lambda}\,,$$

which implies that

$$C = \frac{Q_0}{k\left(\lambda^{-2} - \mathrm{i}\omega/k\right)} = \frac{Q_0\lambda^2}{k}\frac{1 + \mathrm{i}\lambda^2\omega/k}{1 + (\lambda^2\omega/k)^2}\,.$$

To make our lives a little bit easier, let's define $\tilde{Q} = Q_0\lambda^2/k$, $\epsilon = \lambda^2\omega/k$ and $\ell = \sqrt{k/\omega}$ (note that $\epsilon = \lambda^2/\ell^2$). The total solution can then be written

$$w = A\mathrm{e}^{z\sqrt{\mathrm{i}}/\ell} + B\mathrm{e}^{-z\sqrt{\mathrm{i}}/\ell} + \frac{1+\mathrm{i}\epsilon}{1+\epsilon^2}\tilde{Q}\mathrm{e}^{z/\lambda}\,.$$

The condition that $w' \to 0$ as $z \to -\infty$ requires $B = 0$. The condition at $z = 0$ implies

$$u_0 = A + \frac{1+\mathrm{i}\epsilon}{1+\epsilon^2}\tilde{Q}\,,$$

and so

$$A = u_0 - \frac{1+\mathrm{i}\epsilon}{1+\epsilon^2}\tilde{Q}\,.$$

The final solution for $w(z)$ is then

$$w(z) = u_0\mathrm{e}^{z\sqrt{\mathrm{i}}/\ell} + \frac{1+\mathrm{i}\epsilon}{1+\epsilon^2}\tilde{Q}\left(\mathrm{e}^{z/\lambda} - \mathrm{e}^{z\sqrt{\mathrm{i}}/\ell}\right)\,.$$

(c) Finding the real part of $w(z)$ takes some care. We use the fact that

$$\sqrt{\mathrm{i}} = \frac{1}{\sqrt{2}}\left(\mathrm{i}+1\right)\,.$$

This implies that

$$\mathrm{Re}\left[\mathrm{e}^{z\sqrt{\mathrm{i}}/\ell}\right] = \mathrm{e}^{z/\sqrt{2}\ell}\cos(z/\sqrt{2}\ell)\,,$$

and

$$\mathrm{Re}\left[\mathrm{i}\mathrm{e}^{z\sqrt{\mathrm{i}}/\ell}\right] = -\mathrm{e}^{z/\sqrt{2}\ell}\sin(z/\sqrt{2}\ell)$$

Therefore

$$\begin{aligned}\mathrm{Re}\left[w(z)\right] &= u_0\mathrm{e}^{z/\sqrt{2}\ell}\cos(z/\sqrt{2}\ell) \\ &\quad + \frac{\tilde{Q}}{1+\epsilon^2}\left(\mathrm{e}^{z/\lambda} - \mathrm{e}^{z/\sqrt{2}\ell}\cos(z/\sqrt{2}\ell)\right) + \frac{\epsilon\tilde{Q}}{1+\epsilon^2}\mathrm{e}^{z/\sqrt{2}\ell}\sin(z/\sqrt{2}\ell)\,, \\ &= u_0\mathrm{e}^{z/\sqrt{2}\ell}\cos(z/\sqrt{2}\ell) + \frac{\tilde{Q}}{1+\epsilon^2}\left[\mathrm{e}^{z/\lambda} - \mathrm{e}^{z/\sqrt{2}\ell}\Big(\cos(z/\sqrt{2}\ell) - \epsilon\sin(z/\sqrt{2}\ell)\Big)\right].\end{aligned}$$

3 Laplace's equation in a square. Consider Laplace's equation in Cartesian coordinates in (x, y),

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0\,,$$

with the boundary conditions

$$u(x = 0, y) = -1\,, \qquad u(x, y = 0) = 0\,,$$
$$u(x = L, y) = 1\,, \qquad u(x, y = L) = 0\,.$$

A sketch is given below.

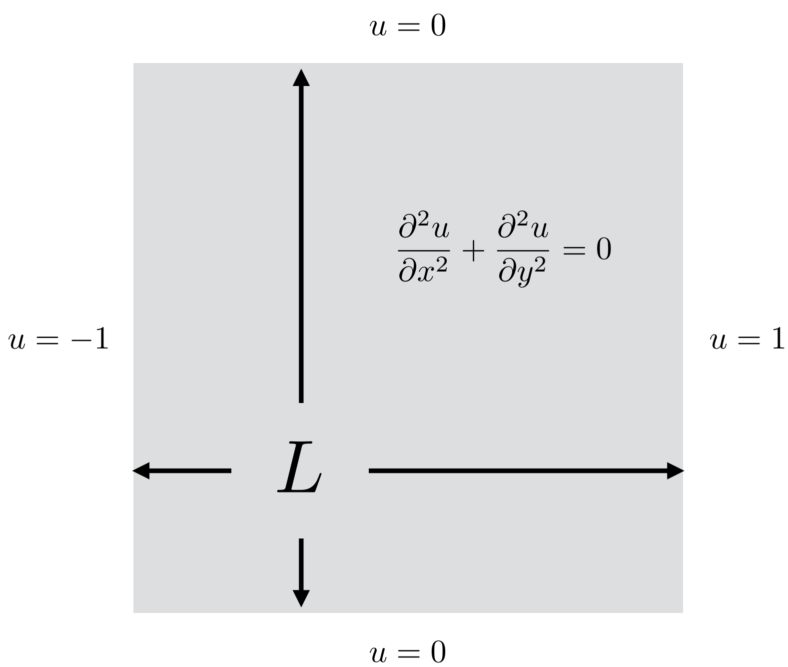

Figure 2: "Sketch" for problem 4.

(a) Use the principle of superposition and separation of variables to find $u(x, y)$ which satisfies the governing equation and all boundary conditions.

(b) What is the solution for $u(x, y)$ when the boundary conditions at $y = 0$ and $y = L$ are both changed to

$$\frac{\partial u}{\partial y} = 0\,?$$

Finding the solution should not require more than a line or two of calculation. *Hint: will the solution depend on y?*

Solution.

(a) We use separation of variables to find u_1 and u_2. First, we substitute $u = f(x)g(y)$ into the differential equation, and divide by $f(x)g(y)$. This yields

$$\frac{f''}{f} = -\frac{g''}{g}.$$

The left side of the equation is a function of x only, while the right side is a function of y. Thus they can only be equal if they are both equal to a constant, which we call λ, and which implies

$$\frac{f''}{f} = -\frac{g''}{g} = \lambda.$$

We then obtain the two equations

$$f'' - \lambda f = 0, \qquad g'' + \lambda g = 0,$$

with the boundary conditions The equation for f is a boundary value problem with inhomogeneous boundary conditions, whereas the equation for g is an eigenvalue problem which determines the permissible values of λ.

For both problems, the g-equation has solutions of the form

$$g = a\sin(\sqrt{\lambda}y) + b\cos(\sqrt{\lambda}y).$$

The boundary condition $g(y=0) = 0$ implies $B = 0$, and $g(y = L) = 0$ implies $\lambda = (n\pi/L)^2$, where n is a positive integer.

We decompose u into

$$u(x,y) = u_1(x,y) + u_2(x,y).$$

The solution u_1 satisfies $u_1 = 1$ at $x = L$ and $u_1 = 0$ on the three other boundaries, while the solution u_2 satisfies $u_2 = -1$ at $x = 0$, and $u_2 = 0$ on the other boundaries.

For u_1, the solution for f_1 which satisfies $f_1 = 0$ at $x = 0$ is

$$f_1 = c_n \sinh(n\pi x/L).$$

The total solution for u_1 is

$$u_1 = \sum_{n=1}^{\infty} A_n \sinh(n\pi x/L)\sin(n\pi y/L).$$

We find the value for A using the boundary condition at $x = L$, which implies

$$1 = \sum_{n=1}^{\infty} A_n \sinh(n\pi)\sin(n\pi y/L)$$

Multiplying by $\sin(n\pi y/L)$ and integrating over the domain implies

$$A_n = \frac{2}{n\pi\sinh(n\pi)}\left(1-\cos(n\pi)\right), \tag{1}$$
$$= \frac{4}{n\pi\sinh(n\pi)} \times \begin{cases} 1 & n \text{ odd}, \\ 0 & n \text{ even} \end{cases} \tag{2}$$

We can write the odd n as $n = 2p-1$, and rewrite this as

$$A_p = \frac{4}{(2p-1)\pi\sinh((2p-1)\pi)}, \qquad \text{for } p = 1, 2, 3, \ldots.$$

Therefore

$$u_1(x,y) = \sum_{p=1}^{\infty} \frac{4\sinh((2p-1)\pi x/L)}{(2p-1)\pi\sinh((2p-1)\pi)} \sin(n\pi y/L).$$

The problem for u_2 is very similar, except that the solution for f_2 is

$$f_2 = d_n \sinh(n\pi(x-L)/L).$$

This solution satisfies $f_2(x=L) = 0$. At $x = 0$, we have

$$-1 = \sum_{n=1}^{\infty} A_n \sinh(-n\pi)\sin(n\pi y/L).$$

The $A_n's$ are therefore identical to before, and we find

$$u_2 = \sum_{p=1}^{\infty} \frac{4\sinh((2p-1)\pi(x-L)/L)}{(2p-1)\pi\sinh((2p-1)\pi)} \sin(n\pi y/L).$$

The total solution, $u = u_1 + u_2$, is then

$$u = \sum_{p=1}^{\infty} \frac{4\sin\left(\frac{(2p-1)\pi y}{L}\right)}{(2p-1)\pi\sinh((2p-1)\pi)} \left[\sinh\left(\frac{(2p-1)\pi x}{L}\right) + \sinh\left(\frac{(2p-1)\pi}{L}(x-L)\right)\right].$$

The solution for $u(x,y)$ is plotted in figure 3.

(b) The hint strongly suggests that the solution will not depend on y. This means we simply need to solve

$$\frac{d^2u}{dx^2} = 0,$$

with $u(0) = -1$ and $u(L) = 1$. The solution is a straight line between -1 and 1, or

$$u = -1 + \frac{2x}{L}.$$

It is easy to check that this satisfies boundary conditions. Alternatively, we could obtain the solution using the procedure in (a). The resulting lengthy calculation would

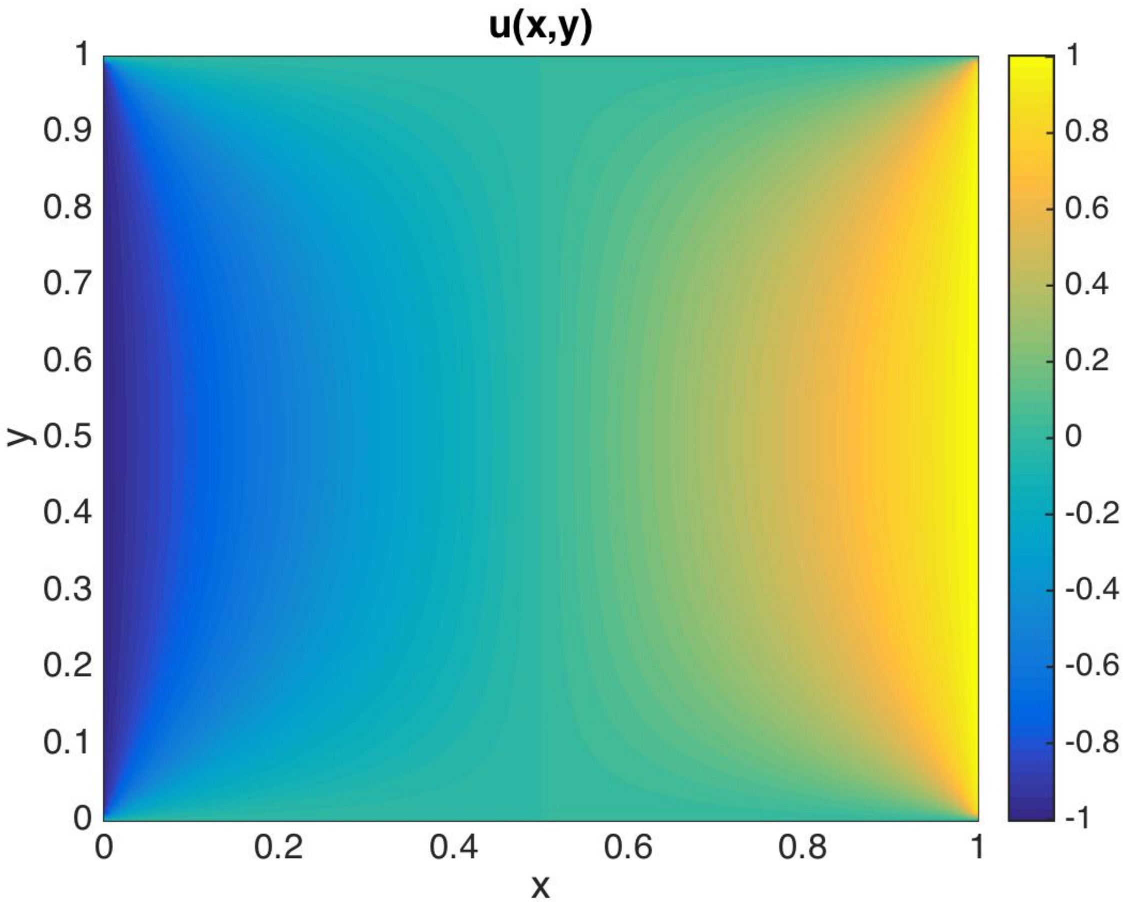

Figure 3: The solution $u(x, y)$ to 3(a) (the second 3).

reveal that the y-modes are cosines, but that the only mode remaining that satisfies the boundary conditions is the one for which $\lambda = 0$; i.e., the one that does not depend on y.

Note that if we find a solution to Laplace's equation which satisfies the boundary conditions, we are guaranteed that this is the single correct solution. Thus for simple problems, finding the solution by educated guess is a legitimate and powerful tool.

4 Laplace's equation outside a disk. Consider Laplace's equation outside the disk with radius a. The domain thus extends from $r = a$ to ∞. Laplace's equation in polar coordinates is

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0\,.$$

The boundary condition at $r = a$ is

$$\frac{\partial u}{\partial r}(r = a, \theta) = 1 + 2\sin\theta\,.$$

The boundary condition as $r \to +\infty$ is

$$\nabla u(r \to \infty, \theta) \to 0\,.$$

A sketch is below.

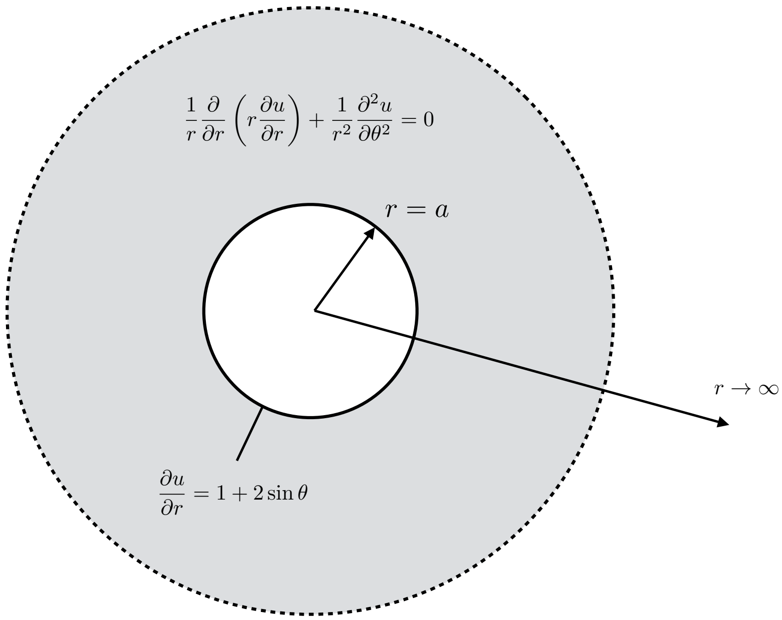

Figure 4: "Sketch" for problem 5.

(a) Solve for $u(r,\theta)$. *Hint: your solution will contain an undeterminable constant. This is because there are two solutions with no θ-dependence.*

Solution. To use separation of variables we propose $u(r,\theta) = f(r)g(\theta)$ and plug this into the governing equation. After multiplying by r^2/fg and rearranging, we obtain

$$\frac{r}{f}\left(rf'\right)' = -\frac{g''}{g} = \lambda\,.$$

The g-equation is the eigenvalue problem. Note that θ goes from $\theta = 0$ to $\theta = 2\pi$, and we have periodic conditions on g such that

$$g(0) = g(2\pi) \qquad \text{and} \qquad g'(0) = g'(2\pi)\,.$$

The g-equation is therefore

$$g'' + \lambda g = 0\,,$$

which has the solution

$$g = A\sin\left(\sqrt{\lambda}\theta\right) + B\cos\left(\sqrt{\lambda}\theta\right)\,.$$

Both are valid solutions under periodic boundary conditions given that $\sqrt{\lambda} = n$, where n is an integer. Notice that $n = 0$ corresponds to the non-trivial solution where g is constant and $u(r, \theta)$ does not depend on θ. Given that $\lambda = n^2$, the equation for f is

$$r^2 f'' + r f' - n^2 f = 0 .$$

When $n > 0$, we can solve this equation by proposing $f = Cr^{\alpha}$. We then have $f' = C\alpha r^{\alpha-1}$ and $f'' = C(\alpha^2 - \alpha) r^{\alpha-2}$, which implies that

$$\alpha^2 - n^2 = 0 ,$$

and $\alpha = \pm n$. Thus, for $n > 0$, we find

$$f = Cr^n + Dr^{-n} .$$

One of our boundary conditions is that $\nabla u \to 0$ as $r \to \infty$. Note that

$$\nabla u = \frac{\partial u}{\partial r} \hat{\mathbf{r}} + \frac{1}{r} \frac{\partial u}{\partial \theta} \hat{\boldsymbol{\theta}} ,$$

where $\hat{\mathbf{r}}$ and $\hat{\boldsymbol{\theta}}$ are unit vectors in the r- and θ-directions, respectively. As a consequence, we must have both that $f' \to 0$ and that $f/r \to 0$ as $r \to \infty$. The solution $f = Cr^n$ is incompatible with this condition and, therefore, we must have $C = 0$.

However, this is not the whole solution, since there are valid solutions which do not depend on θ, corresponding to the case $n = 0$. In this case, the equation for f is

$$r f'' + f' = 0 .$$

To solve this equation, we propose the form $f' = Er^{\alpha}$, which implies that $\alpha = -1$. Thus

$$f' = \frac{E}{r} ,$$

and

$$f = E \ln r + F .$$

This comprises the part of the solution independent of θ. Both $f = F$ and $f = E \ln r$ satisfy the conditions that $f' \to 0$ and $f/r \to 0$ as $r \to \infty$. The total solution for $u(r, \theta)$ is then

$$u(r, \theta) = F + E \ln r + \sum_{n=1}^{\infty} r^{-n} \left(A_n \sin(n\theta) + B_n \cos(n\theta) \right) .$$

Thus, we can calculate $\partial u / \partial r$,

$$\frac{\partial u}{\partial r} = \frac{E}{r} + \sum_{n=1}^{\infty} \left(-n r^{-n-1} \right) \left(A_n \sin(n\theta) + B_n \cos(n\theta) \right) ,$$

and at $r = a$ we must have

$$\frac{\partial u}{\partial r}\bigg|_{r=a} = 1 + 2\sin(\theta) = \frac{E}{a} + \sum_{n=1}^{\infty}\left(-na^{-n-1}\right)\left(A_n \sin(n\theta) + B_n \cos(n\theta)\right) .$$

Next, we multiply both sides by $\sin(\theta)$ and integrate from $\theta = 0$ to $\theta = 2\pi$. We then find that

$$2\int_0^{2\pi} \sin^2(\theta)\, \mathrm{d}\theta = -a^{-2}A_1 \int_0^{2\pi} \sin^2(\theta)\, \mathrm{d}\theta ,$$

where all the other terms corresponding to B_n and A_n for $n \neq 1$ have disappeared, a consequence of the orthogonality of sines and cosines. We then find that

$$A_1 = -2a^2 .$$

The other part of the initial condition can be obtained simply by integrating from 0 2π (one might think of this as multiplying by $\cos(0 * \theta)$ and integrating). All the sines and cosines disappear, and we are left with

$$2\pi = 2\pi\frac{E}{a} , \qquad \text{which implies} \qquad E = a .$$

The total solution for $u(r, \theta)$ is therefore

$$u(r, \theta) = F + a\ln r - \frac{2a^2}{r}\sin(\theta) .$$

The given boundary conditions do not permit evaluation of F – this is the best we can do! Notice that this form satisfies the boundary conditions and the governing equation.