Homework 3

Due April 29, 2015.

1 Review of inhomogeneous equations. Give the general solution to the following ODEs:

$$
\begin{aligned}
&(a) \qquad y'' - 4y = \mathrm{e}^x \,, \\
&(b) \qquad y'' + 4y = \sin(2x) \,, \\
&(c) \qquad x^2 y'' + 2xy' + y = 0 \,, \\
&(d) \qquad x^2 y'' + 3xy' + y = 0 \,.
\end{aligned}
$$

Hint: one of these questions involves resonance and one has a repeated root. In these cases, consider solutions of the form $x\,\mathrm{e}^{kx}$ *or* $x^\beta \ln x$ *(where* k *can be imaginary).*

2 Fourier sine and cosine series'.

(a) Find the Fourier sine series of

$$f(x) = \mathrm{e}^x \,,$$

on the interval $0 \le x \le L$.

(b) Find the Fourier cosine series of $f(x)$ on the same interval.

Hint: consider the real and imaginary parts of the integral $\int_0^L \mathrm{e}^{x + \mathrm{i} n\pi x/L}\, \mathrm{d}x$.

3 Fourier Series. For a periodic function on the interval $-\pi \le x \le \pi$, the Fourier representation is

$$f(x) = a_0 + \sum_{n=1} a_n \cos(nx) + b_n \sin(nx) \,.$$

For the functions in (a) and (b), find the coefficients a_n and b_n.

$$\text{(a)} \qquad f(x) = \begin{cases} 0 & \text{for} \quad x \le -\pi/2 \quad \text{and} \quad x \ge \pi/2 \\ 1 & \text{for} \quad -\pi/2 < x < \pi/2 \end{cases} \tag{1}$$

$$\text{(b)} \qquad f(x) = \begin{cases} x + \pi & \text{for} \quad x \le 0 \\ \pi - x & \text{for} \quad x > 0 \end{cases} \tag{2}$$

4 The whacked wave equation. Consider the wave equation,

$$\frac{\partial^2 u}{\partial t^2} - c^2 \frac{\partial^2 u}{\partial x^2} = 0\,,$$

with fixed ends, so that

$$u(0,t) = u(L,t) = 0\,,$$

and two initial conditions: zero initial displacement,

$$u(x,0) = 0\,,$$

and with an initial impulsive whacking velocity of

$$\frac{\partial u}{\partial t}(x,0) \quad = \quad \begin{cases} \delta & \text{for} \quad L/2 - \delta \le x \le L/2 + \delta\,, \\ 0 & \qquad\qquad \text{otherwise}\,, \end{cases}$$

where $0 < \delta < L/2$. Find $u(x,t)$ using separation of variables.

5 Laplace's equation in a 60° wedge. Consider Laplace's equation in a circular wedge with radius 1 in polar coordinates (r,θ), where $0 \le r \le 1$ and $0 \le \theta \le \pi/3$. Laplace's equation is

$$\nabla^2 u = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial^2 \theta} = 0\,,$$

and the boundary conditions are

$$u(r,0) = u(r,\pi/3) = 0\,,$$

and

$$u(1,\theta) = h(\theta)\,.$$

We also have the condition that u is bounded at $r = 0$, or that $|u(0,\theta)| < \infty$. Find $u(r,\theta)$ using separation of variables.