Homework 3

Due April 29, 2015.

1 Review of inhomogeneous equations. Give the general solution to the following ODEs:

$$\begin{aligned}
&(a) \qquad y'' - 4y = \mathrm{e}^{x}\,, \\
&(b) \qquad y'' + 4y = \sin(2x)\,, \\
&(c) \qquad x^2 y'' + 2xy' + y = 0\,, \\
&(d) \qquad x^2 y'' + 3xy' + y = 0\,.
\end{aligned}$$

Hint: one of these questions involves resonance and one has a repeated root. In these cases, consider solutions of the form $x\,\mathrm{e}^{kx}$ or $x^{\beta}\ln x$ (where k can be imaginary).

Solution.

(a) The homogeneous solution to (a) solves $y'' - 4y = 0$ and can be found by plugging in $y_h = A\mathrm{e}^{kx}$ into the governing equation. This yields that $k = \pm 2$, and that the homogeneous solution is

$$y_h(x) = A\mathrm{e}^{2x} + B\mathrm{e}^{-2x}\,.$$

Since the forcing term, e^{x}, does not match any of the homogeneous solutions, it is not resonant. An exponential function is particularly simple in that we can guess a particular solution $y_p(x) = C\mathrm{e}^{x}$, plug this into the governing equation, and solve for C. This yields

$$y_p'' - 4y_p = C\mathrm{e}^{x}\Big(1-4\Big) = \mathrm{e}^{x}\,.$$

This implies that $C = -1/3$, and the general solution is

$$y(x) = y_h + y_p = A\mathrm{e}^{2x} + B\mathrm{e}^{-2ex} - \tfrac{1}{3}\mathrm{e}^{x}\,.$$

(b) The homogeneous solution is

$$y_h(x) = A\sin(2x) + B\cos(2x)\,.$$

It is apparent that the forcing term takes the form of one of the homogeneous solutions, $\sin(2x)$. This is called “resonant forcing”. First, we use the identity

$$\mathrm{e}^{i\theta} = \cos(\theta) + \mathrm{i}\sin(\theta)\,,$$

to write the governing equation as

$$y'' + 4y = \frac{\mathrm{i}}{2}\left(\mathrm{e}^{-2\mathrm{i}x} - \mathrm{e}^{2\mathrm{i}x}\right).$$

Next, we propose a particular solution of the form $y_p = Cx\mathrm{e}^{kx}$. We then have

$$y_p = Cx\mathrm{e}^{kx}, \tag{1}$$
$$y_p' = C\mathrm{e}^{kx}\left(kx + 1\right), \tag{2}$$
$$y_p'' = C\mathrm{e}^{kx}\left(k^2x + 2k\right). \tag{3}$$

Putting this into the governing equation yields

$$Cx\mathrm{e}^{kx}\left(k^2 + 4\right) + 2kC\mathrm{e}^{kx} = \frac{\mathrm{i}}{2}\left(\mathrm{e}^{-2\mathrm{i}x} - \mathrm{e}^{2\mathrm{i}x}\right).$$

The first term involving $Cx\mathrm{e}^{kx}$ can only match the right side if $k^2 + 4 = 0$, or if $k = \pm 2\mathrm{i}$. This leaves us with two equations, one for $k = +2\mathrm{i}$ and one for $k = -2\mathrm{i}$, which allow us to choose two constants, C_+ and C_-, to match the right hand side. In other words,

$$4\mathrm{i}C_+\mathrm{e}^{2\mathrm{i}x} = -\tfrac{\mathrm{i}}{2}\mathrm{e}^{2\mathrm{i}x},$$

and

$$-4\mathrm{i}C_-\mathrm{e}^{-2\mathrm{i}x} = \tfrac{\mathrm{i}}{2}\mathrm{e}^{-2\mathrm{i}x}.$$

These equations imply that both C_+ and C_- are

$$C_+ = C_- = -\tfrac{1}{8}.$$

The particular solution is then

$$y_p = -\tfrac{1}{8}x\left(\mathrm{e}^{2\mathrm{i}x} + \mathrm{e}^{-2\mathrm{i}x}\right) = -\tfrac{1}{4}x\cos(2x),$$

where for the last step we again use Euler's identity.

(c) This is an "equidimensional" equation. To solve this type of equations we propose a solution of the form $y = Ax^\alpha$. We then have

$$y = Ax^\alpha, \tag{4}$$
$$y' = A\alpha x^{\alpha-1}, \tag{5}$$
$$y'' = A\alpha(\alpha - 1)x^{\alpha-2}. \tag{6}$$

Plugging these into the governing equation yields the "characteristic equation" for α,

$$\alpha^2 - \alpha + 2\alpha + 1 = \alpha^2 + \alpha + 1 = 0.$$

We can find the solution using the quadratic equation,

$$\alpha = \tfrac{1}{2}\Big(-1 \pm \sqrt{1-4}\Big) = -\frac{1}{2} \pm \frac{\mathrm{i}\sqrt{3}}{2}\,.$$

Note that the properties of logarithms imply that

$$x^{\imath\beta} = \mathrm{e}^{\mathrm{i}\beta\ln x} = \cos\left(\beta \ln x\right) + \mathrm{i}\sin\left(\beta\ln x\right)\,.$$

So, our two solutions are

$$y_1 = Ax^{(-1+\mathrm{i}\sqrt{3})/2} \qquad \text{and} \qquad y_2 = Bx^{(-1-\mathrm{i}\sqrt{3})/2}\,,$$

and the general solution is

$$y = Ax^{(-1+\mathrm{i}\sqrt{3})/2} + Bx^{(-1-\mathrm{i}\sqrt{3})/2} = x^{-1/2}\left[C\cos\left(\tfrac{\sqrt{3}}{2}\ln x\right) + D\sin\left(\tfrac{\sqrt{3}}{2}\ln x\right)\right].$$

(d) If we propose a solution of the form $y = Ax^\alpha$, the characteristic equation for α is

$$\alpha^2 - \alpha + 3\alpha + 1 = \Big(\alpha + 1\Big)^2 = 0\,.$$

This is the case of a repeated root, where both the roots of α are -1. We thus have only found one solution, $y_1 = Ax^{-1} = A/x$. To find the second solution, we guess a solution of the form

$$y_2 = Bx^\beta \ln x\,.$$

We then have

$$y_2 = Bx^\beta \ln x\,, \tag{7}$$
$$y_2' = B\beta x^{\beta-1}\ln x + x^{\beta-1}\,, \tag{8}$$
$$y_2'' = B\beta(\beta-1)x^{\beta-2}\ln x + B\beta x^{\beta-2} + B(\beta-1)x^{\beta-2}\,. \tag{9}$$

Plugging this into the governing equation yields

$$0 = Bx^{\beta-2}\ln x\Big(\beta^2 - \beta + 3\beta + 1\Big) + Bx^{\beta-2}\Big(2\beta - 1 + 3\Big) \tag{10}$$
$$= Bx^{\beta-2}\ln x\Big(\beta+1\Big)^2 + Bx^{\beta-2}\Big(2\beta+2\Big)\,. \tag{11}$$

From this we deduce (as we might have expected) that $\beta = -1$ and $y_2 = B\ln x/x$. The total solution is then

$$y = Ax^{-1} + Bx^{-1}\ln x\,.$$

2 Fourier sine and cosine series'.

(a) Find the Fourier sine series of

$$f(x) = \mathrm{e}^x ,$$

on the interval $0 \leq x \leq L$.

(b) Find the Fourier cosine series of $f(x)$ on the same interval.

Hint: consider the real and imaginary parts of the integral $\int_0^L \mathrm{e}^{x+\mathrm{i}n\pi x/L}\,\mathrm{d}x$.

Solution.

1. The Fourier sine series of a function $f(x)$ on the interval 0 to L is defined as

$$f_s(x) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi x}{L}\right) ,$$

where the coefficients A_n are given by

$$A_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)\,\mathrm{d}x .$$

Using $f(x) = \mathrm{e}^x$ yields the integral

$$A_n = \frac{2}{L}\int_0^L \mathrm{e}^x \sin\left(\frac{n\pi x}{L}\right)\,\mathrm{d}x .$$

One way to evaluate this integral is to note that

$$\mathrm{Im}\left[\mathrm{e}^{\mathrm{i}n\pi x/L}\right] = \sin\left(\frac{n\pi x}{L}\right) .$$

The operation $\mathrm{Im}[\cdot]$ means that we take the imaginary part of whatever is inside the brackets. Therefore, we can rewrite the integral for A_n as

$$A_n = \frac{2}{L}\,\mathrm{Im}\left[\int_0^L \mathrm{e}^{x+\mathrm{i}n\pi x/L}\,\mathrm{d}x\right] .$$

The integral in the brackets is easily found,

$$\int_0^L \mathrm{e}^{x+\mathrm{i}n\pi x/L}\,\mathrm{d}x = \frac{1}{1+\mathrm{i}n\pi/L}\mathrm{e}^{x(1+\mathrm{i}n\pi/L)}\Big|_0^L , \tag{12}$$

$$= \frac{1}{1+\mathrm{i}n\pi/L}\left(\mathrm{e}^L\mathrm{e}^{\mathrm{i}n\pi} - 1\right) . \tag{13}$$

Note that $\mathrm{e}^{\mathrm{i}n\pi} = -1$ when n is odd and 1 when n is even, which means we can write $\mathrm{e}^{\mathrm{i}n\pi} = (-1)^n$. Also,

$$\frac{1}{1+\mathrm{i}n\pi/L} = \frac{L\,(L-\mathrm{i}n\pi)}{L^2+(n\pi)^2} ,$$

which can be shown by first multiplying top and bottom by L, and then by $(L - \mathrm{i}n\pi)$. We therefore find that

$$\mathrm{Im}\left[\int_0^L \mathrm{e}^{x+\mathrm{i}n\pi x/L}\,\mathrm{d}x\right] = -\frac{Ln\pi}{L^2+(n\pi)^2}\left(\mathrm{e}^L(-1)^n - 1\right),$$

and that

$$A_n = \frac{2n\pi}{L^2+(n\pi)^2}\left(1 - \mathrm{e}^L(-1)^n\right).$$

The Fourier sine series of e^x on the interval $x = 0$ to $x = L$ is therefore

$$f_s(x) = \sum_{n=1}^{\infty} \frac{2n\pi}{L^2+(n\pi)^2}\left(1 - \mathrm{e}^L(-1)^n\right)\sin\left(\frac{n\pi x}{L}\right).$$

2. The cosine series, fortunately, can be calculated very easily using previous results. The cosine series is defined

$$f_c(x) = \sum_{n=0}^{\infty} B_n \cos\left(\frac{n\pi x}{L}\right),$$

where B_0 is defined by

$$B_0 = \frac{1}{L}\int_0^L f(x)\,\mathrm{d}x,$$

and the rest of the B_n for $n > 0$ are defined by

$$B_n = \frac{2}{L}\int_0^L f(x)\cos\left(\frac{n\pi x}{L}\right).$$

Using Euler's identity we can show, similar to part (a), that

$$B_n = \frac{2}{L}\mathrm{Re}\left[\int_0^L f(x)\cos\left(\frac{n\pi x}{L}\right)\right].$$

We already calculated the integral; all we have to do is take the real part, instead of the imaginary part. The real part is

$$\mathrm{Re}\left[\int_0^L f(x)\cos\left(\frac{n\pi x}{L}\right)\right] = \frac{L^2}{L^2+(n\pi)^2}\left(\mathrm{e}^L(-1)^n - 1\right).$$

Therefore, the B_n (for $n > 0$are given by

$$B_n = \frac{2L}{L^2+(n\pi)^2}\left(\mathrm{e}^L(-1)^n - 1\right).$$

For $n = 0$, we have

$$B_0 = \frac{1}{L}\int_0^L \mathrm{e}^x\,\mathrm{d}x = \mathrm{e}^x\Big|_0^L = \frac{1}{L}\left(\mathrm{e}^L - 1\right).$$

The Fourier cosine series of $f(x)$ is therefore

$$f_c(x) = \frac{\mathrm{e}^L - 1}{L} + \sum_{n=0}^{\infty} \frac{2L}{L^2+(n\pi)^2}\left(\mathrm{e}^L(-1)^n - 1\right)\cos\left(\frac{n\pi x}{L}\right).$$

3 Fourier Series. For a periodic function on the interval $-\pi \leq x \leq \pi$, the Fourier representation is

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos(nx) + b_n \sin(nx) \,.$$

For the functions in (a) and (b), find the coefficients a_n and b_n.

$$\text{(a)} \qquad f(x) = \begin{cases} 0 & \text{for} \quad x \leq -\pi/2 \quad \text{and} \quad x \geq \pi/2 \\ 1 & \text{for} \quad -\pi/2 < x < \pi/2 \end{cases} \tag{14}$$

$$\text{(b)} \qquad f(x) = \begin{cases} x + \pi & \text{for} \quad x \leq 0 \\ \pi - x & \text{for} \quad x > 0 \end{cases} \tag{15}$$

Solution. The coefficients a_n and b_n are given by the following formulas

$$n = 0\,, \qquad a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x)\, \mathrm{d}x\,, \tag{16}$$

$$n > 1\,, \qquad a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos(nx)\, \mathrm{d}x\,, \tag{17}$$

$$n > 1\,, \qquad b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin(nx)\, \mathrm{d}x\,. \tag{18}$$

(a) The coefficient a_0 is given by

$$a_0 = \frac{1}{2\pi} \int_{-L}^{L} f(x)\, \mathrm{d}x = \frac{1}{2\pi} \int_{-\pi/2}^{\pi/2} 1\, \mathrm{d}x = \tfrac{1}{2}\,.$$

The coefficients a_n for $n > 1$ are given by

$$a_n = \frac{1}{\pi} \int_{-\pi/2}^{\pi/2} \cos(nx)\, \mathrm{d}x = \frac{2}{n\pi} \sin\left(\frac{n\pi}{2}\right) = \frac{2}{n\pi} \begin{cases} (-1)^{\frac{n-1}{2}} & \text{for } n \text{ odd} \\ 0 & \text{for } n \text{ even}\,. \end{cases}$$

The coefficients b_n are 0. We know this to be true because $f(x)$ is an even function around $x = 0$; in other words, $f(x) = f(-x)$ for $0 < x < \pi$. When $f(x)$ is even, it does not project onto the sine coefficients b_n. Conversely, when x is odd, it does not project onto a_n. We can also show that the b_n are zero in this case by direct computation, since

$$b_n = \frac{1}{\pi} \int_{-\pi/2}^{\pi/2} \sin(nx)\, \mathrm{d}x = -\frac{1}{n\pi} \cos(nx)\Big|_{-\pi/2}^{\pi/2} = 0.$$

Substituting $n = 2\pi - 1$ (since a_n only contributes to the Fourier series when n is odd), the Fourier series representation can be written

$$f(x) = \tfrac{1}{2} + \sum_{p=1}^{\infty} \frac{2(-1)^{p-1}}{(2p-1)\pi} \cos\Big[(2p-1)x\Big]\,.$$

(b) The coefficient a_0 is given by

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{0} x + \pi \, \mathrm{d}x + \frac{1}{2\pi}\int_{0}^{\pi} \pi - x \, \mathrm{d}x = \frac{1}{\pi}\left[\tfrac{1}{2}x^2 + \pi x\right]_{-\pi}^{0} = \tfrac{1}{2}\pi \, .$$

The coefficients a_n are

$$a_n = \frac{1}{\pi}\int_{-\pi}^{0} (x+\pi)\cos(nx) \, \mathrm{d}x + \frac{1}{\pi}\int_{0}^{\pi} (\pi - x)\cos(nx) \, \mathrm{d}x \, .$$

We can simplify this expression by noting that both integrals are equal to one another. This can be shown by substituting $x' = -x$ into the second integral, for example. This substitution implies that $\mathrm{d}x = -\,\mathrm{d}x$. Using the fact that $\cos(\theta) = \cos(-\theta)$, we then find that

$$I = \frac{1}{\pi}\int_{0}^{\pi} (\pi - x)\cos(nx) \, \mathrm{d}x \, , \tag{19}$$

$$= \frac{1}{\pi}\int_{0}^{-\pi} \left(\pi + x'\right)\cos(-nx')(-\,\mathrm{d}x') \, , \tag{20}$$

$$= \frac{1}{\pi}\int_{-\pi}^{0} \left(\pi + x'\right)\cos(nx') \, \mathrm{d}x' \, . \tag{21}$$

In the final step, the negative sign incurred by the swapping of limits cancels the negative sign in front of $-\,\mathrm{d}x'$. We can therefore write the expression for a_n as a single integral, which can be evaluated using integration by parts. We find

$$a_n = \frac{2}{\pi}\int_{-\pi}^{0} (x+\pi)\cos(nx) \, \mathrm{d}x \, , \tag{22}$$

$$= \frac{2}{n}\sin(nx)\Big|_{-\pi}^{0} + \frac{2x}{\pi n}\sin(nx)\Big|_{-\pi}^{0} - \frac{2}{\pi n}\int_{-\pi}^{0} \sin(nx) \, \mathrm{d}x \, , \tag{23}$$

$$= \frac{2}{\pi n^2}\left(1 - \cos(n\pi)\right) \, , \tag{24}$$

$$= \frac{4}{\pi n^2}\begin{cases} 1 & n \text{ odd} \, , \\ 0 & n \text{ even} \, . \end{cases} \tag{25}$$

The b_n are zero, for the same reason in (a). We can show this with direct integration as well. If we introduce $n = 2p - 1$, we can write the Fourier series as

$$f(x) = \tfrac{1}{2}\pi + \sum_{p=1}^{\infty} \frac{4}{\pi(2p-1)^2}\cos\left[(2p-1)x\right] .$$

4 The whacked wave equation. Consider the wave equation,

$$\frac{\partial^2 u}{\partial t^2} - c^2\frac{\partial^2 u}{\partial x^2} = 0 \, ,$$

with fixed ends, so that

$$u(0,t) = u(L,t) = 0 \, ,$$

and two initial conditions: zero initial displacement,

$$u(x,0) = 0\,,$$

and with an initial impulsive whacking velocity of

$$\frac{\partial u}{\partial t}(x,0) \quad = \quad \begin{cases} \delta & \text{for} \quad L/2-\delta \le x \le L/2+\delta\,, \\ 0 & \qquad\qquad \text{otherwise}\,, \end{cases}$$

where $0 < \delta < L/2$. Find $u(x,t)$ using separation of variables.

Solution. We use separation of variables by proposing $u(x,t) = f(x)g(t)$. Plugging this form into the governing equation and multiplying by $1/c^2fg$ yields

$$\frac{g''}{c^2g} = \frac{f''}{f}\,.$$

Applying the usual argument that the groupings to the right and left of the equals sign can only equal each other if they are each separately equal to a constant, we obtain the two ODE's,

$$f'' + \lambda f = 0\,,$$

and

$$g'' + c^2\lambda g = 0\,,$$

where we have defined the separation constant λ. The spatial boundary conditions on $u(x,t)$ imply that $f(0) = f(L) = 0$. These homogeneous boundary conditions on f imply that the f-equation provides the eigenvalue problem which determines λ. Assuming that $\lambda > 0$, which must be true for f to have non-trivial solutions, the solutions to the f-equation are

$$f = A\sin\left(\sqrt{\lambda}x\right) + B\cos\left(\sqrt{\lambda}x\right)\,.$$

The condition that $f(0) = 0$ implies that $B = 0$. The condition that $f(L) = 0$ implies then that

$$0 = A\sin\left(\sqrt{\lambda}L\right)\,.$$

If $A \neq 0$, the boundary condition can only be satisfied if $\sin\left(\sqrt{\lambda}L\right) = 0$, which occurs when

$$\sqrt{\lambda}L = n\pi\,, \qquad \text{which implies that} \qquad \lambda = \left(\frac{n\pi}{L}\right)^2\,.$$

The solutions to the f-equation are therefore

$$f_n = A_n\sin\left(\frac{n\pi x}{L}\right)\,.$$

Correspondingly, the solutions to the g-equation are

$$g = C\sin\left(\frac{n\pi ct}{L}\right) + D\cos\left(\frac{n\pi ct}{L}\right)\,.$$

Since $u(x, t=0) = 0$, we must have $D = 0$. The general solution for $u(x,t)$ is then

$$u(x,t) = \sum_{n=1} A_n \sin\left(\frac{n\pi x}{L}\right) \sin\left(\frac{n\pi ct}{L}\right) .$$

Note that this means

$$\frac{\partial u}{\partial t} = \sum_{n=1} \frac{A_n n\pi c}{L} \sin\left(\frac{n\pi x}{L}\right) \cos\left(\frac{n\pi ct}{L}\right) .$$

The coefficients A_n are determined by the initial condition, which implies that

$$\frac{\partial u}{\partial t}(x, t=0) = \phi(x) = \sum_{n=1}^{\infty} \frac{A_n n\pi c}{L} \sin\left(\frac{n\pi x}{L}\right) ,$$

where $\phi(x)$ is

$$\phi(x) \quad = \quad \begin{cases} \delta & \text{for} \quad L/2-\delta \leq x \leq L/2+\delta\,, \\ 0 & \qquad\qquad \text{otherwise}\,, \end{cases}$$

We obtain an expression for the A_n by multiplying the expression for $\phi(x)$ by $\sin(m\pi x/L)$ and integrating from $x = 0$ to $x = L$. This yields the expression

$$\int_0^L \phi(x) \sin\left(\frac{m\pi x}{L}\right)\,\mathrm{d}x = \frac{A_m m\pi c}{L} \int_0^L \sin^2\left(\frac{m\pi x}{L}\right)\,\mathrm{d}x\,.$$

Inserting the form for $\phi(x)$, we then find that the A_n are given by

$$\begin{aligned}
A_n &= \frac{2}{n\pi c} \int_0^L \phi(x) \sin\left(\frac{n\pi x}{L}\right)\,\mathrm{d}x\,, && (26)\\
&= \frac{2}{n\pi c} \int_{L/2-\delta}^{L/2+\delta} \delta \sin\left(\frac{n\pi x}{L}\right)\,\mathrm{d}x\,, && (27)\\
&= \frac{2\delta L}{c(n\pi)^2} \Big[-\cos(n\pi x/L) \Big]_{L/2-\delta}^{L/2+\delta}, && (28)\\
&= \frac{2\delta L}{c(n\pi)^2} \left[\cos\left(n\pi\left[\tfrac{1}{2}+\delta/L\right]\right) - \cos\left(n\pi\left[\tfrac{1}{2}-\delta/L\right]\right) \right]. && (29)
\end{aligned}$$

With this rather cumbersome formula, we have the full solution for $u(x,t)$. Figure 1 shows the displacement $u(x,t)$ and the velocity $\partial u/\partial t$ at a few times after the initial whack.

5 Laplace's equation in a 60° wedge. Consider Laplace's equation in a circular wedge with radius 1 in polar coordinates (r, θ), where $0 \leq r \leq 1$ and $0 \leq \theta \leq \pi/3$. Laplace's equation is

$$\nabla^2 u = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial^2 \theta} = 0\,,$$

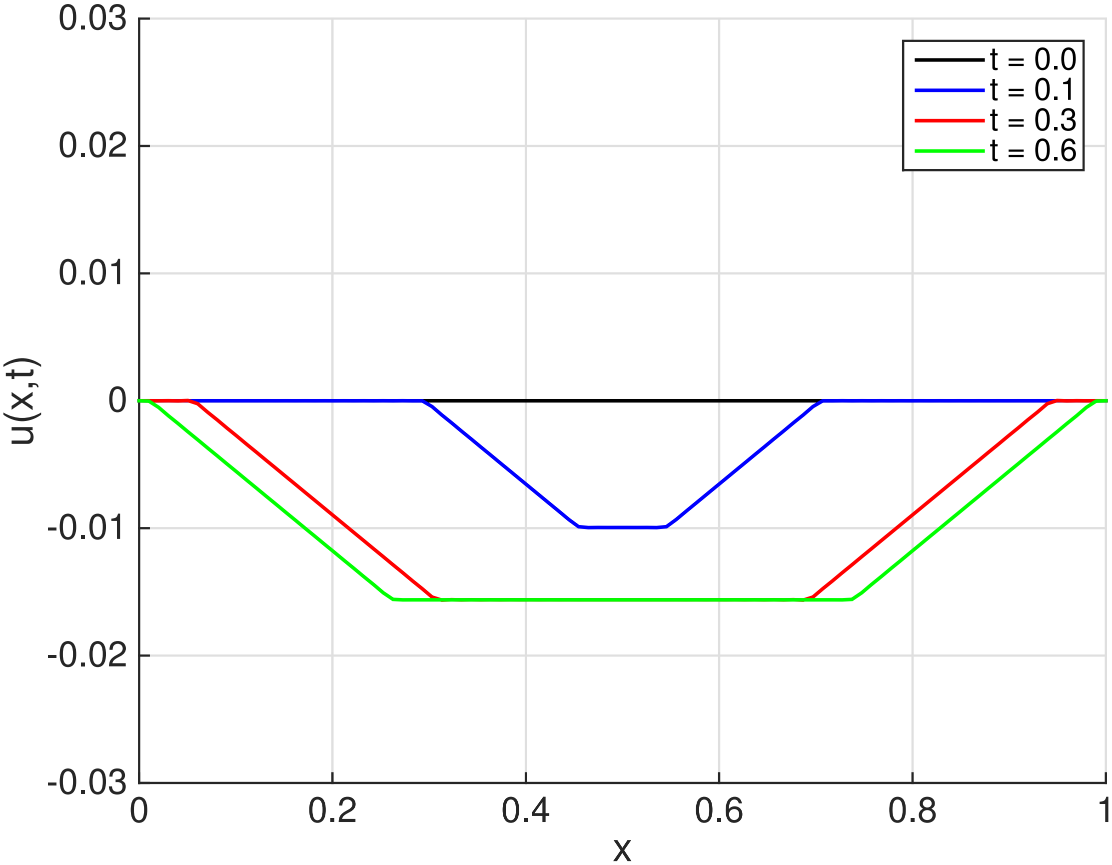

Figure 1: A whacked string. The solution $u(x,t)$ for problem 4 at various times t.

and the boundary conditions are

$$u(r,0) = u(r,\pi/3) = 0\,,$$

and

$$u(1,\theta) = h(\theta)\,.$$

We also have the condition that u is bounded at $r = 0$, or that $|u(0,\theta)| < \infty$. Find $u(r,\theta)$ using separation of variables.

Solution. We use separation of variables by proposing that $u = f(r)g(\theta)$. Similar to the disk problem in homework 2, we substitute this form into the governing equation and multiply by r^2/fg. This yields

$$\frac{r}{f}\left(rf'\right)' = -\frac{g''}{g} = \lambda\,,$$

where we have defined a separation constant λ. The equation for g is then

$$g'' + \lambda g = 0\,.$$

The boundary conditions on $g(\theta)$ are $g(0) = g(\pi/3) = 0$, which follow from the boundary conditions on $u(r,\theta)$. As a consequence, we must find oscillatory solutions, and $\lambda > 0$. The general solution for g is

$$g = A \sin\left(\sqrt{\lambda}\theta\right) + B\cos\left(\sqrt{\lambda}\theta\right) .$$

The condition that $g(0) = 0$ implies that $B = 0$. The condition at $\theta = \pi/3$ implies that

$$0 = A\sin\left(\sqrt{\lambda}\pi/3\right) .$$

This can only be zero when $\sin\left(\sqrt{\lambda}\pi/3\right) = 0$ which occurs when $\sqrt{\lambda}\pi/3 = n\pi$, where n is an integer. Thus we find $\lambda = (3n)^2$, and the θ-modes are

$$g_n = A_n \sin(3n\theta) .$$

Note that we can take $n > 0$ without loss of generality, because the modes for $n < 0$ are identical to the modes for $n > 0$ (because $\sin(3\theta) = -\sin(-3\theta)$). The equation for $f(r)$ becomes

$$r^2 f'' + rf' + 9n^2 f = 0 .$$

We solve this equation by proposing $f = Cr^{\alpha}$. This yields a characteristic equation for α,

$$\alpha^2 = 9n^2 \qquad \text{which implies} \qquad \alpha = \pm 3n .$$

The solution for $f(r)$ is therefore

$$f = Cr^{3n} + Dr^{-3n} .$$

When $r \to 0$, the function Dr^{-3n} diverges to $+\infty$. Thus, this solution violates the condition that $|u(r=0)| < \infty$, and we must have $D = 0$. Reconstructing the total solution for u, we have

$$u(r,\theta) = \sum_{n=1}^{\infty} A_n r^{3n} \sin(3n\theta) .$$

We now can apply the condition at $r = 1$. This implies that

$$h(\theta) = \sum_{n=1}^{\infty} A_n \sin(3n\theta) .$$

To find the coefficients A_n, we project this condition on the modes $\sin(3m\theta)$ by multiplying by $\sin(3m\theta)$ and integrating from $\theta = 0$ to $\theta = \pi/3$. This yields

$$A_m = \frac{6}{\pi}\int_0^{\pi/3} h(\theta)\sin(3m\theta)\,\mathrm{d}\theta .$$