Homework 4

Due May 13, 2015.

1 Heat equation in a square. Solve the heat equation for $u(x, y, t)$ in the square $0 < x < L$, $0 < y < L$. On the boundaries, assume the normal derivative vanishes, that is $\partial u/\partial n = \hat{\mathbf{n}} \cdot \nabla u = 0$ (where $\hat{\mathbf{n}}$ is a normal vector to the boundary pointing out of the domain), which implies

$$\frac{\partial u}{\partial x} = 0 \qquad \text{at } x = 0, L\,,$$

and

$$\frac{\partial u}{\partial y} = 0 \qquad \text{at } y = 0, L\,.$$

Take a general initial condition $u(x, y, t = 0) = \phi(x, y)$.

2 Wave equation in a rectangle. The wave equation in Cartesian coordinates is

$$\frac{\partial^2 u}{\partial t^2} = c^2 \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right) .$$

We consider solving this equation in a rectangular domain, where $0 < x < L$ and $0 < y < H$. On the boundaries we use the condition $u = 0$, which implies

$$u(0, y, t) = u(L, y, t) = u(x, 0, t) = u(x, H, t) = 0\,.$$

Proceeding as in class, find the solution satisfying the initial conditions

$$u(x, y, 0) = x(L - x)\, y(H - y), \qquad \frac{\partial u}{\partial t}(x, y, 0) = 0\,.$$

3 Wave equation on a circular membrane. Consider the wave equation on a circular membrane of radius a. The wave equation in polar coordinates is,

$$\frac{\partial^2 u}{\partial t^2} = c^2 \left[\frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial u}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 u}{\partial \theta^2} \right] .$$

Use the boundary condition $\partial u/\partial n = 0$ on the boundaries, which implies

$$\frac{\partial u}{\partial r} = 0 \qquad \text{at } r = a\,.$$

Answer the following:

1. Use separation of variables to derive three equations, one depending on t, one depending on r, and one depending on θ. Write down the solution to the t-dependent equation and the θ-dependent equation.

2. Your r-dependent equation will take the form

$$r^2 \frac{d^2 f}{dr^2} + r \frac{df}{dr} + \left(\lambda^2 r^2 - n^2\right) f = 0,$$

   where λ is an eigenvalue and n is an integer. If we introduce the substitution $z = \lambda r$ and divide by z^2, we obtain Bessel's equation from class. The solution bounded at $r = 0$ is therefore $f(r) = AJ_n(\lambda r)$, where A is a constant and J_n is the Bessel function of the first kind. The eigenvalues λ_{mn} are determined by the boundary condition at $r = a$ and requires finding the zeros of the Bessel functions (or the derivatives of the Bessel functions). For now, don't worry about finding the λ_{mn}.

   Now, rewrite Bessel's equation in Sturm–Liouville form and, using the results of Sturm-Liouville theory, derive the orthogonality relation for the functions $J_n(\lambda_{mn} r)$ over the interval $(0, a)$.

3. Using the orthogonality relation for the functions $J_n(\lambda a)$ over $(0, a)$, write the general solution to the wave equation when the initial conditions are

$$\frac{\partial u}{\partial t}(r, \theta, t = 0) = 0, \qquad \text{and} \qquad u(r, \theta, t = 0) = \phi(r, \theta).$$

4. Take $n = 1$ in Bessel's equation and change variable to x, where $x = r/a$. Write down the transformed version of Bessel's equation and the corresponding Rayleigh quotient (notice that the boundary terms vanish from the Rayleigh quotient, leaving only terms that involve integrals).

5. Now consider the test function $F(x) = 2x - x^2$. First, taking into account that $x = r/a$, confirm that $F(x)$ satisfies the condition on the r-dependent solution $f(r)$. Next, use the Rayleigh quotient for Bessel's equation to generate an estimate for the first zero of $J_1'(x)$, which corresponds to $\lambda_1 a$. [*Hint: Your answer should be a fraction which is quite close to the exact result* 1.84118378134054...].