Homework 4

Due May 13, 2015.

1 Heat equation in a square. Solve the heat equation for $u(x,y,t)$ in the square $0 < x < L$, $0 < y < L$. On the boundaries, assume the normal derivative vanishes, that is $\partial u/\partial n = \hat{\mathbf{n}} \cdot \nabla u = 0$ (where $\hat{\mathbf{n}}$ is a normal vector to the boundary pointing out of the domain), which implies

$$\frac{\partial u}{\partial x} = 0 \qquad \text{at } x = 0, L\,,$$

and

$$\frac{\partial u}{\partial y} = 0 \qquad \text{at } y = 0, L\,.$$

Take a general initial condition $u(x,y,t=0) = \phi(x,y)$.

Solution. The heat equation is

$$\frac{\partial u}{\partial t} = k\nabla^2 u = k\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right)\,.$$

With the given initial and boundary conditions, we can solve the heat equation using separation of variables. We first propose $u = S(x,y)T(t)$ and insert this into the heat equation. After dividing by kST, we obtain

$$\frac{T'}{kT} = \frac{\nabla^2 S}{S} = -\lambda\,,$$

where we have indicated that, because either side is a functions of t or (x,y), but not both, the only way they can be equal is if they are both equal to a constant. We denote this constant "$-\lambda$" so the t-equation yields exponentially decreasing solutions. The t-equation is

$$T' + k\lambda T = 0\,,$$

and the solution is

$$T = A\mathrm{e}^{-k\lambda t}\,.$$

We turn to the equation in (x,y), which is

$$\nabla^2 S + \lambda S = 0\,.$$

We separate variables again by proposing $S(x,y) = f(x)g(y)$. and dividing by $f(x)g(y)$. This yields

$$\frac{f''}{f} + \lambda = -\frac{g''}{g} = \mu\,,$$

where we have noted the existence of a second separation constant, μ, which arises for the same reason as the first. Note that the boundary conditions on u imply that

$$f'(0) = f'(L) = 0\,, \qquad \text{and} \qquad g'(0) = g'(L) = 0\,.$$

The equation for $g(y)$ is

$$g'' + \mu g = 0\,.$$

This equation has sine and cosine solutions; however, the only solution that can satisfy the condition $g'(0) = 0$ is cosine; therefore

$$g = B\cos\left(\sqrt{\mu}y\right)\,.$$

Applying the condition $g'(L) = 0$ implies that μ must be $\mu_n = (n\pi/L)^2$, where n is an integer. The solution for $g(y)$ is therefore

$$g = B_n \cos\left(\frac{n\pi y}{L}\right)\,.$$

We move on to the x-equation, which is

$$f'' + (\lambda - \mu_n)f = 0\,.$$

Remember that we found μ_n in the equation for g. We leave it as μ_n here to save some effort in rewriting it. As for $g(y)$, the only solution for $f(x)$ that satisfies $f'(0) = 0$ is

$$f = C\cos\left(\sqrt{\lambda - \mu_n}x\right)\,.$$

Then, the condition that $f'(L) = 0$ implies that the combination $\lambda - \mu_n$ must be equal to

$$\lambda - \mu_n = \left(\frac{m\pi}{L}\right)^2\,,$$

where m is a second, different integer from n. This further implies that

$$\lambda_{mn} = \mu_n + \left(\frac{m\pi}{L}\right)^2 = \left(\frac{n\pi}{L}\right)^2 + \left(\frac{m\pi}{L}\right)^2\,,$$

and that the solution for $f(x)$ is

$$f(x) = C_m \cos\left(\frac{m\pi x}{L}\right)\,.$$

We can thus construct the total solution for $u(x, y)$,

$$u(x, y, t) = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} A_{mn}\cos\left(\frac{m\pi x}{L}\right)\cos\left(\frac{n\pi y}{L}\right)\mathrm{e}^{-k\lambda_{mn}t}\,,$$

where $\lambda_{mn} = \left(\frac{n\pi}{L}\right)^2 + \left(\frac{m\pi}{L}\right)^2$. We apply the initial condition to determine the constant A_{mn}. Taking $t = 0$ in our expression for $u(x, y, t)$, this implies

$$\phi(x, y) = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} A_{mn}\cos\left(\frac{m\pi x}{L}\right)\cos\left(\frac{n\pi y}{L}\right)\,.$$

The above expression is an infinite sum over *two* indicies n and m. We isolate the x-index by multiplying one of the spatial modes, which can be written

$$\cos\left(\frac{p\pi x}{L}\right).$$

Multiplying by this and integrating over x gives

$$\int_0^L \phi \cos\left(\frac{p\pi x}{L}\right)\,\mathrm{d}x = \left[\sum_{n=0}^{\infty} A_{pn}\cos\left(\frac{n\pi y}{L}\right)\right]\int_0^L \cos^2\left(\frac{p\pi x}{L}\right)\,\mathrm{d}x\,.$$

We now isolate the y-index by multilplying by a y-mode, which for this problem is identical to the x-modes. We write this mode as $\cos(q\pi y/L)$. So, multiplying by $\cos(q\pi y/L)$ and integrating over y yields

$$\int_0^L\int_0^L \phi\cos\left(\frac{p\pi x}{L}\right)\cos\left(\frac{q\pi y}{L}\right)\,\mathrm{d}x\,\mathrm{d}y = A_{pq}\left[\int_0^L \cos^2\left(\frac{q\pi y}{L}\right)\,\mathrm{d}y\right]\left[\int_0^L \cos^2\left(\frac{p\pi y}{L}\right)\,\mathrm{d}x\right].$$

This is a general expression for A_{pq}, and solves the problem. We can go a step further, however, because we can evaluate the integrals. When $p > 0$ and $q > 0$ each integral is equal to $L/2$, and thus their product is $L^2/4$. When $p = 0$ but $q > 0$, or when $q = 0$ and $p > 0$, one of the integrals is L and thus their product is $L^2/2$. Finally, when $p = q = 0$, both integrals are L and their product is L^2. The general solution to the heat equation in a square with the given boundary conditions is therefore

$$u(x,y,t) = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} A_{mn}\cos\left(\frac{m\pi x}{L}\right)\cos\left(\frac{n\pi y}{L}\right)\exp\left[-k\left[\left(\tfrac{n\pi}{L}\right)^2+\left(\tfrac{m\pi}{L}\right)^2\right]\right],$$

where the A_{mn} are given by

$$A_{mn} = \int_0^L\int_0^L \phi\cos\left(\frac{m\pi x}{L}\right)\cos\left(\frac{n\pi y}{L}\right)\,\mathrm{d}x\,\mathrm{d}y\begin{cases}\frac{4}{L^2} & \text{when} \quad n>0\,, m>0\\ \frac{2}{L^2} & \text{when} \quad \Big(n=0\,, m>0\Big)\ \text{or}\ \Big(m=0\,, n>0\Big)\\ \frac{1}{L^2} & \text{when} \quad n=m=0\,.\end{cases}$$

2 Wave equation in a rectangle. The wave equation in Cartesian coordinates is

$$\frac{\partial^2 u}{\partial t^2} = c^2\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right).$$

We consider solving this equation in a rectangular domain, where $0 < x < L$ and $0 < y < H$. On the boundaries we use the condition $u = 0$, which implies

$$u(0,y,t) = u(L,y,t) = u(x,0,t) = u(x,H,t) = 0\,.$$

Proceeding as in class, find the solution satisfying the initial condition

$$u(x,y,0) = x(L-x)\,y(H-y), \qquad \frac{\partial u}{\partial t}(x,y,0) = 0\,.$$

Solution. The procedure for the wave equation is essentially identical to the heat equation, except that the t-equation has oscillatory solutions rather than exponentially decaying ones. We begin by separating variables with $u = T(t)S(x,y)$, inserting this into the wave equation, dividing by c^2ST, and introducing a separation constant. Going through the motions, we find

$$\frac{T''}{c^2T} = \frac{\nabla^2 S}{S} = -\kappa^2 .$$

Above we have *defined* the separation constant suggestively as κ^2. Of course, we can use whatever form for the separation constant that we like. κ^2 seems like a nice choice, because then the solution to the T equation is

$$T = A\cos\left(c\kappa t\right) .$$

In the above we have eliminated the other solution, $\sin(c\kappa t)$, because it cannot satisfy the initial condition $\partial u/\partial t = 0$ at $t = 0$. The $S(x,y)$ equation is

$$\nabla^2 S + \kappa^2 S = 0 .$$

The procedure for $S(x,y)$ is identical to that for the heat equation, so we move more quickly through this solution. We separate variables by proposing $S(x,y) = f(x)g(y)$, inserting it into the governing equation, dividing by fg, and identifying a separation constant, which we define λ^2. This implies

$$\frac{f''}{f} + \kappa^2 = -\frac{g''}{g} = \lambda^2 .$$

The boundary conditions on f and g are

$$f(0) = f(L) = g(0) = g(H) = 0 .$$

The y-equation is $g'' + \lambda^2 g = 0$, and the boundary conditions imply that the eigenfunction solutions are $g = B_n \sin(n\pi y/H)$ and that $\lambda = n\pi/H$. The x-equation is

$$f'' + \left(\kappa^2 - \lambda^2\right) f = 0 .$$

Similar to the y-equation, the solutions satisfying both boundary conditions have the form $f(x) = C_m \sin(m\pi x/L)$, and that

$$\kappa^2 - \lambda^2 = \left(\frac{n\pi}{H}\right)^2 \qquad \text{which implies} \qquad \kappa_{mn} = \sqrt{\left(\tfrac{m\pi}{L}\right)^2 + \left(\tfrac{n\pi}{H}\right)^2} .$$

The total solution for $u(x,y,t)$ can then be written

$$u(x,y,t) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} A_{mn} \sin\left(\frac{m\pi x}{L}\right) \sin\left(\frac{n\pi y}{H}\right) \cos\left(c\kappa_{mn} t\right) ,$$

where

$$\kappa_{mn} = \sqrt{\left(\tfrac{m\pi}{L}\right)^2 + \left(\tfrac{n\pi}{H}\right)^2} .$$

To find the coefficients A_{mn} we apply the initial condition, which implies

$$x(L-x)\,y(H-y) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} A_{mn}\sin\left(\frac{m\pi x}{L}\right)\sin\left(\frac{n\pi y}{H}\right)$$

As before, we isolate the coefficients by multiplying by a spatial mode and integrating. We take a more accelerated route than in the discussion for Problem 1 by multiplying by

$$\sin\left(\frac{p\pi x}{L}\right)\sin\left(\frac{q\pi y}{H}\right)$$

and integrating over both x and y. This yields

$$\begin{aligned}\int_0^H\int_0^L x(L-x)\,y(H-y)\sin\left(\frac{p\pi x}{L}\right)\sin\left(\frac{q\pi y}{H}\right)\,\mathrm{d}x\,\mathrm{d}y \\ = A_{pq}\left[\int_0^H \sin^2\left(\frac{q\pi y}{H}\right)\,\mathrm{d}y\right]\left[\int_0^L \sin^2\left(\frac{p\pi x}{L}\right)\,\mathrm{d}x\right].\end{aligned}$$

Note that we do not have the cases $p=0$ or $q=0$ as in the heat equation. Also note that the integral on the left side can be performed over x and y separately, or in other words,

$$\begin{aligned}\int_0^H\int_0^L x(L-x)\,y(H-y)\sin\left(\frac{p\pi x}{L}\right)\sin\left(\frac{q\pi y}{H}\right)\,\mathrm{d}x\,\mathrm{d}y \\ = \left[\int_0^H y(H-y)\sin\left(\frac{q\pi y}{H}\right)\,\mathrm{d}y\right]\left[\int_0^L x(L-x)\sin\left(\frac{p\pi x}{L}\right)\,\mathrm{d}x\right].\end{aligned}$$

Therefore the general form for A_{pq} is

$$A_{pq} = \frac{4}{HL}\left[\int_0^H y(H-y)\sin\left(\frac{q\pi y}{H}\right)\,\mathrm{d}y\right]\left[\int_0^L x(L-x)\sin\left(\frac{p\pi x}{L}\right)\,\mathrm{d}x\right].$$

Now we calculate the y-integral. We use integration by parts, where in terms of the rule $\int v\,\mathrm{d}w = wv - \int w\,\mathrm{d}v$ we define

$$\begin{aligned} v &= y(H-y)\,, & w &= -\frac{H}{q\pi}\cos\left(\frac{q\pi y}{H}\right)\,, \\ \mathrm{d}v &= H - \tfrac{1}{2}y\,, & \mathrm{d}w &= \sin\left(\frac{q\pi y}{H}\right)\,. \end{aligned}$$

Notice that $v = y(H-y)$ vanishes at both 0 and H, and therefore the term vw term does not contribute to the integration by parts. Therefore,

$$\int_0^H y(H-y)\sin\left(\frac{q\pi y}{H}\right)\,\mathrm{d}y = -\int v\,\mathrm{d}w = \int_0^H \frac{H}{q\pi}\left(H-\tfrac{1}{2}y\right)\cos\left(\frac{q\pi y}{H}\right)\,\mathrm{d}y\,.$$

We iterate again with

$$v = H - \tfrac{1}{2}y\,, \qquad w = \left(\frac{H}{q\pi}\right)^2 \sin\left(\frac{q\pi y}{H}\right)\,,$$

$$\mathrm{d}v = -\tfrac{1}{2}\,, \qquad \mathrm{d}w = \frac{H}{q\pi}\cos\left(\frac{q\pi y}{H}\right)\,.$$

Notice that this time, w vanishes at 0 and H and again the term vw does not contribute to the integral. We thus get

$$\begin{aligned}\int_0^H \frac{H}{q\pi}\left(H - \tfrac{1}{2}y\right)\cos\left(\frac{q\pi y}{H}\right)\,\mathrm{d}y &= \frac{1}{2}\int_0^H \left(\frac{H}{q\pi}\right)^2 \sin\left(\frac{q\pi y}{H}\right)\,\mathrm{d}y\,,\\ &= -\frac{1}{2}\left(\frac{H}{q\pi}\right)^3 \cos\left(\frac{q\pi y}{H}\right)\Big|_0^H\,,\\ &= \frac{1}{2}\left(\frac{H}{q\pi}\right)^3\Big(1-\cos(q\pi)\Big)\,.\end{aligned}$$

The x-integral is exactly the same with H replaced by L. Therefore

$$\int_0^L x(L-x)\sin\left(\frac{p\pi x}{L}\right)\,\mathrm{d}x = \frac{1}{2}\left(\frac{L}{p\pi}\right)^3\Big(1-\cos(p\pi)\Big)\,.$$

The general solution for $u(x,y,t)$ can therefore be written

$$u(x,y,t) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} A_{mn}\sin\left(\frac{m\pi x}{L}\right)\sin\left(\frac{n\pi y}{H}\right)\cos\left(c\kappa_{mn}t\right)\,,$$

where

$$\kappa_{mn} = \sqrt{\left(\tfrac{n\pi}{H}\right)^2 + \left(\tfrac{n\pi}{H}\right)^2}\,.$$

and

$$A_{mn} = \frac{(HL)^2}{(mn\pi^2)^3}\Big(1-\cos(m\pi)\Big)\Big(1-\cos(n\pi)\Big)\,.$$

It is possible to simplify this expression further by noting that $1-\cos(m\pi)$ is either 0 or 2, but we leave it in this form here, which is perfectly valid.

3 Wave equation on a circular membrane. Consider the wave equation on a circular membrane of radius a. The wave equation in polar coordinates is,

$$\frac{\partial^2 u}{\partial t^2} = c^2\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}\right]\,.$$

Use the boundary condition $\partial u/\partial n = 0$ on the boundaries, which implies

$$\frac{\partial u}{\partial r} = 0 \qquad \text{at } r = a\,.$$

Answer the following:

1. Use separation of variables to derive three equations, one depending on t, one depending on r, and one depending on θ. Write down the solution to the t-dependent equation and the θ-dependent equation.

2. Your r-dependent equation will take the form

$$r^2\frac{\mathrm{d}^2 f}{\mathrm{d}r^2} + r\frac{\mathrm{d}f}{\mathrm{d}r} + \left(\lambda^2 r^2 - n^2\right) f = 0\,,$$

   where λ is an eigenvalue and n is an integer. If we introduce the substitution $z = \lambda r$ and divide by z^2, we obtain Bessel's equation from class. The solution bounded at $r = 0$ is therefore $f(r) = AJ_n(\lambda r)$, where A is a constant and J_n is the Bessel function of the first kind. The eigenvalues λ_{mn} are determined by the boundary condition at $r = a$ and requires finding the zeros of the Bessel functions (or the derivatives of the Bessel functions). For now, don't worry about finding the λ_{mn}.

   Now, rewrite Bessel's equation in Sturm–Liouville form and, using the results of Sturm-Liouville theory, derive the orthogonality relation for the functions $J_n(\lambda_{mn}r)$ over the interval $(0, a)$.

3. Using the orthogonality relation for the functions $J_n(\lambda a)$ over $(0, a)$, write the general solution to the wave equation when the initial conditions are

$$\frac{\partial u}{\partial t}(r, \theta, t = 0) = 0\,, \qquad \text{and} \qquad u(r, \theta, t = 0) = \phi(r, \theta)\,.$$

4. Take $n = 1$ in Bessel's equation and change variable to x, where $x = r/a$. Write down the transformed version of Bessel's equation and the corresponding Rayleigh quotient (notice that the boundary terms vanish from the Rayleigh quotient, leaving only terms that involve integrals).

5. Now consider the test function $F(x) = 2x - x^2$. First, taking into account that $x = r/a$, confirm that $F(x)$ satisfies the condition on the x-dependent solution $f(x)$. Next, use the Rayleigh quotient for Bessel's equation to generate an estimate for the first zero of $J_1'(x)$, which corresponds to $\lambda_1 a$. [*Hint: Your answer should be a fraction which is quite close to the exact result* 1.84118378134054...].

Solution.

1. Initial steps

The wave equation can be written

$$\frac{\partial^2 u}{\partial t^2} = c^2\nabla^2 u\,.$$

To separate variables, we first propose that $u = S(r,\theta)T(t)$, substitute this form into the wave equation, and divide by $c^2 ST$. We obtain

$$\frac{T''}{c^2 T} = \frac{\nabla^2 S}{S} = -\lambda^2 \,,$$

where we have made the additional step in observing that, in this form, the wave equation has a solution only if both left and right sides are equal to a constant, and defined that separation constant "$-\lambda^2$". The time dependent equation is

$$T'' + c^2\lambda^2 T = 0 \,,$$

and the solution is

$$T = a\cos(c\lambda t) + b\sin(c\lambda t) \,.$$

The time dependent equation is simple. But solving the spatial equation requires yet another application of separation of variables. We introducing the form for ∇^2, the equation for $S(r,\theta)$ becomes

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial S}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 S}{\partial \theta^2} + \lambda^2 S = 0 \,.$$

To make further progress, we propose $S = f(r)g(\theta)$. Substituting this into the S-equation yields

$$\frac{g}{r}\left(rf'\right)' + \frac{f}{r^2}g'' + \lambda^2 fg = 0 \,.$$

Next, we divide by fg and multiply by r^2. Moving the g-dependent part onto the other side of the equation, we find

$$\frac{r}{f}\left(rf'\right)' + \lambda^2 r^2 = -\frac{g''}{g}$$

The left side is a function of r and the right side is a function of θ; therefore they must be equal to a constant. Because we expect periodic solutions for $g(\theta)$, we denote this constant n^2 (we can later verify that n is indeed an integer). We thus write

$$\frac{r}{f}\left(rf'\right)' + \lambda^2 r^2 = -\frac{g''}{g} = n^2 \,,$$

and the g-equation, or θ-dependent equation, is

$$g'' + n^2 g = 0 \,,$$

which has the solutions

$$g = c\cos(n\theta) + d\sin(n\theta) \,.$$

Both of these solutions satisfy periodic conditions at $\theta = 0, 2\pi$ if n is an integer. The r-dependent equation is

$$r\left(rf'\right)' + \left(\lambda^2 r^2 - n^2\right)f = 0 \,.$$

or

$$r^2 f'' + r f' + \left(\lambda^2 r^2 - n^2\right) f = 0$$

In summary, the three equations are

$$\begin{aligned} t &: \qquad T'' + c^2\lambda^2 T = 0\,, \\ \theta &: \qquad g'' + n^2 g = 0\,, \\ r &: \qquad r^2 f'' + r f' + \left(\lambda^2 r^2 - n^2\right) f = 0\,. \end{aligned}$$

The solution to the t-dependent and θ-dependent equation are

$$T = a\cos(c\lambda t) + b\sin(c\lambda t)\,.$$

and

$$g = c\cos(n\theta) + d\sin(n\theta)\,.$$

2. Bessel's equation as a Strum-Liouville problem

Using $\mathrm{d}/\mathrm{d}r$ for derivatives, Bessel's equation is

$$r^2\frac{\mathrm{d}^2 f}{\mathrm{d}r^2} + r\frac{\mathrm{d}f}{\mathrm{d}r} + \left(\lambda^2 r^2 - n^2\right) f = 0\,.$$

If we divide by r and note that

$$\frac{\mathrm{d}}{\mathrm{d}r}\left(r\frac{\mathrm{d}f}{\mathrm{d}r}\right) = r\frac{\mathrm{d}^2 f}{\mathrm{d}r^2} + \frac{\mathrm{d}f}{\mathrm{d}r}\,,$$

we can write Bessel's equation in the form

$$\frac{\mathrm{d}}{\mathrm{d}r}\left(r\frac{\mathrm{d}f}{\mathrm{d}r}\right) - \frac{n^2}{r} f + \lambda^2 r f = 0\,.$$

This is the Strum-Liouville form with $p = r$, $q = -n^2/r$, $\sigma = r$, and boundaries at $r = 0$ and $r = a$. The Sturm-Liouville eigenvalue is λ^2.

This is a "singular" Sturm-Liouville eigenvalue problem because q goes to infinity as $r \to 0$. However, for this particular, problem, we can still use the results of Sturm-Liouville. Note that "n" is a *parameter* in the problem. Thus, we know there will be an infinity number of solutions *for every n*. The eigenvalue relation in lectures and book was given as

$$\int_0^a \phi_m \phi_p \sigma\,\mathrm{d}r = 0\,,$$

where $m \neq p$, so that ϕ_m and ϕ_p are *different* eigenfunctions. The solutions to Bessel's equation are $f = J_n\left(\lambda_{mn} r\right)$. Plugging in these eigenfunctions and using $\sigma = r$, the orthogonality relation is

$$\int_0^a J_n(\lambda_{mn} r) J_n(\lambda_{pn} r) r\,\mathrm{d}r = 0\,,$$

when $m \neq p$.

Note that n is constant. Thus, for example, when $n = 1$, two eigenfunctions of the corresponding Bessel eigenvalue problem are $J_1(\lambda_{11} r)$ and $J_1(\lambda_{21} r)$. Thus for a given n, the solutions have the same form – J_n – while the argument of J_n changes. This situation is much like that for the eigenproblem

$$y'' + y = 0\,, \qquad y(0) = y(2\pi) = 0\,.$$

In this case, every eigenfunction is a sine function with a different argument; for example, $\sin(nx)$ and $\sin(2nx)$. In the Bessel function case, the m values of λ_{nm} corresponding to each n-eigenproblem must be determined numerically (in contrast to the simple case with sines and cosines, where the arguments are integers).

3. A general solution

The solution to the t-problem is

$$T = a\cos(c\lambda t) + b\sin(c\lambda t)\,.$$

The initial condition

$$\frac{\partial u}{\partial t} = 0$$

implies that $b = 0$. The solution to the θ-problem is $g = c\cos(n\theta) + d\sin(n\theta)$ and the solution to the r-problem is $f = AJ_n(\lambda_{mn} r)$. Thus the total solution for $u(r,\theta,t)$ is

$$u(r,\theta,t) = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} J_n\left(\lambda_{mn} r\right)\cos\left(c\lambda_{mn} t\right)\Big(A_{mn}\sin(n\theta) + B_{mn}\cos(n\theta)\Big)\,.$$

Applying the initial condition for $u(r,\theta,t=0)$ implies

$$\phi(r,\theta) = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} J_n\left(\lambda_{mn} r\right)\Big(A_{mn}\sin(n\theta) + B_{mn}\cos(n\theta)\Big)\,.$$

Similar to problems 1 and 2, we multiply by modes in θ and r to isolate the m and n. If we multiply by $\sin(p\theta)$ and integrate from $\theta = 0$ to $\theta = 2\pi$, we obtain

$$\int_0^{2\pi} \phi\sin(p\theta)\,\mathrm{d}\theta = \sum_{m=0}^{\infty} \pi A_{mp} J_p\left(\lambda_{mp} r\right)\,.$$

Next we multiply by $rJ_p(\lambda_{qp} r)$ and integrate from 0 to a. This multiplication allows us to use the orthogonality condition for Bessel functions shown in part 2 to isolate the q^{th} mode from the sum over m. We find

$$\int_0^a \int_0^{2\pi} \phi\sin(p\theta) J_p\left(\lambda_{pq} r\right) r\,\mathrm{d}\theta\,\mathrm{d}r = \pi\int_0^a rJ_p^2\left(\lambda_{qp} r\right)\,\mathrm{d}r A_{qp}\,,$$

or, in terms of m and n

$$A_{mn} = \frac{\int_0^a \int_0^{2\pi} \phi \sin(n\theta) J_n\left(\lambda_{mn} r\right) r \, \mathrm{d}\theta \, \mathrm{d}r}{\pi \int_0^a r J_n^2\left(\lambda_{mn} r\right) \, \mathrm{d}r}.$$

Notice that the formula for the B_{mn} is identical with $\cos(n\theta)$ swapped out for $\sin(n\theta)$ – except for $n = 0$, since in this case the θ integral produces a factor of 2π rather than just π. Therefore,

$$B_{mn} = \frac{\int_0^a \int_0^{2\pi} \phi \cos(n\theta) J_n\left(\lambda_{mn} r\right) r \, \mathrm{d}\theta \, \mathrm{d}r}{\pi \int_0^a r J_n^2\left(\lambda_{mn} r\right) \, \mathrm{d}r}.$$

for $n > 0$, and

$$B_{m0} = \frac{\int_0^a \int_0^{2\pi} \phi J_n\left(\lambda_{mn} r\right) r \, \mathrm{d}\theta \, \mathrm{d}r}{2\pi \int_0^a r J_n^2\left(\lambda_{mn} r\right) \, \mathrm{d}r}.$$

With A_{mn} and B_{mn}, and

$$u(r,\theta,t) = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} J_n\left(\lambda_{mn} r\right) \cos\left(c\lambda_{mn} t\right) \Big(A_{mn} \sin(n\theta) + B_{mn} \cos(n\theta)\Big),$$

we have solved the problem.

Transformed Bessel's equation and the Rayleigh quotient

Bessel's equation with $n = 1$ is

$$\frac{\mathrm{d}}{\mathrm{d}r}\left(r \frac{\mathrm{d}f}{\mathrm{d}r}\right) - \frac{1}{r} f + \lambda^2 r f = 0.$$

The transformation $r = ax \implies x = r/a$ implies that $\mathrm{d}x = \mathrm{d}r/a$, so that $\mathrm{d}x/\mathrm{d}r = 1/a$, and

$$\frac{\mathrm{d}}{\mathrm{d}r} = \frac{\mathrm{d}}{\mathrm{d}x} \frac{\mathrm{d}x}{\mathrm{d}r} = \frac{1}{a} \frac{\mathrm{d}}{\mathrm{d}x}.$$

Thus in terms of x we have

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(x \frac{\mathrm{d}f}{\mathrm{d}x}\right) - \frac{1}{x} f + (\lambda a)^2 x f = 0.$$

The boundary condition at $r = a$ is now applied at $x = (r = a)/a = 1$, and therefore the boundary condition is $\mathrm{d}f/\mathrm{d}x = 0$ at $x = 1$. With the test function $F(x) = 2x - x^2$, we have

$$\frac{\mathrm{d}F}{\mathrm{d}x} = 2 - 2x,$$

and therefore $\mathrm{d}F/\mathrm{d}x(x = 1) = 0$. At $x = 0$, $F(x)$ is bounded, which satisfies the condition on J_1. In fact, $F(x)$ was chosen so it matches the fact that $J_1(x = 0) = 0$, which means it will make for a particularly good test function.

The general Sturm-Liouville form for the Rayleigh quotient for a domain between $x = 0$ and $x = 1$, in terms of a trial function F, is

$$R = \frac{-pF\frac{\mathrm{d}F}{\mathrm{d}x}\Big|_0^1 + \int_0^1 p\left(\frac{\mathrm{d}F}{\mathrm{d}x}\right)^2 - qF^2\,\mathrm{d}x}{\int_0^1 F^2\sigma\,\mathrm{d}x}\,.$$

For $p = x$, $q = -1/x$, and $\sigma = x$, we have that $R \leq (\lambda a)^2$, and the Rayleigh quotient gives an estimate for quantity $(\lambda a)^2$. Note that the boundary terms disappear because $p = 0$ at $x = 0$ and $\mathrm{d}f/\mathrm{d}x = 0$ at $x = 1$. Therefore the Rayleigh quotient can be written

$$R = \frac{\int_0^1 x\left(\frac{\mathrm{d}F}{\mathrm{d}x}\right)^2 + \frac{1}{x}F^2\,\mathrm{d}x}{\int_0^1 xF^2\,\mathrm{d}x}\,.$$

We use the test function $F(x) = 2x - x^2$. Notice that $\mathrm{d}F/\mathrm{d}x = 2 - 2x$, and that

$$F^2 = x^4 - 4x^3 + 4x^2\,,$$

and that

$$\left(\frac{\mathrm{d}F}{\mathrm{d}x}\right)^2 = 4x^2 - 8x + 4\,.$$

The Rayleigh quotient becomes

$$\begin{aligned} R &= \frac{\int_0^1 5x^3 - 12x^2 + 8x\,\mathrm{d}x}{\int_0^1 x^5 - 4x^4 + 4x^3\,\mathrm{d}x}\,, \\ &= \frac{\frac{5}{4} - 4 + 4}{\frac{1}{6} - \frac{4}{5} + 1}\,, \\ &= \frac{75}{22}\,. \end{aligned}$$

Notice that $\lambda_{11}a$ is the first zero the derivative of the first Bessel function, since the condition at $r = a$ requires that

$$\frac{\mathrm{d}}{\mathrm{d}r}J_1(\lambda_{11}r)\Big|_{r=a} = 0\,,$$

Since the Rayleigh quotient gives an estimate for $(\lambda_1 a)^2$, this implies that

$$\lambda_{11}a \approx \sqrt{\frac{75}{22}} = 1.846\,.$$

The actual value of the first zero of $J_1'(x)$ is 1.8411... which means that $\sqrt{75/22}$ is off by just 0.28%.