Homework 5

Due May 20, 2015.

1 Legendre polynomials on a tidally locked planet. The steady heat distribution in a solid sphere is governed by Laplace's equation in spherical coordinates,

$$\nabla^2 u = \frac{1}{\rho^2}\frac{\partial}{\partial\rho}\left(\rho^2\frac{\partial u}{\partial\rho}\right) + \frac{1}{\rho^2\sin\phi}\frac{\partial}{\partial\phi}\left(\sin\phi\frac{\partial u}{\partial\phi}\right) + \frac{1}{\rho^2\sin^2\phi}\frac{\partial^2 u}{\partial\theta^2} = 0\,.$$

In this equation, ρ is the radial coordinate extending from $\rho = 0$ at the center of the planet to $\rho = R$ at the surface. The coordinate ϕ is called colatitude and lies in the range $(0, \pi)$, with 0 being at the North Pole and π at the South Pole. θ is the longitudinal angle, which goes from 0 to 2π. Consider the following:

1. Separate variables by writing $u = S(\rho, \phi)q(\theta)$. Solve the equation for θ.

2. Now assume that u does not depend on θ. This assumption means that we are looking for "axisymmetric" solutions. Separate variables again by writing $S(\rho, \phi) = f(\rho)g(\phi)$ and show that $g(\phi)$ satisfies

$$\frac{\mathrm{d}}{\mathrm{d}\phi}\left(\sin\phi\frac{\mathrm{d}g}{\mathrm{d}\phi}\right) + \mu\sin\phi\, g = 0\,,$$

   where μ is an eigenvalue. The solutions to this Sturm-Liouville equation are called Legendre polynomials. There are two solutions: only one of them is bounded at both $\phi = 0$ and $\phi = \pi$, which happens for the eigenvalue $\mu = n(n+1)$, where $n = 0, 1, 2...$ is an integer. Denote this solution $g_n(\phi) = P_n(\cos\phi)$, where P_n is a polynomial. The first three P_n are

$$\begin{aligned} P_0 &= 1\,, \\ P_1 &= \cos\phi\,, \\ P_2 &= \tfrac{1}{2}\left(3\cos^2\phi - 1\right) = \tfrac{1}{4}\left(3\cos 2\phi + 1\right)\,. \end{aligned}$$

3. Observing that the Legendre equation is a Sturm-Liouville eigenvalue problem, write down the orthogonality relation satisfied by the Legendre polynomials. [*Note: it is not necessary to derive the orthogonality relation from scratch; instead simply write down the results of Sturm-Liouville theory as they apply to this particular problem.*]

4. Solve the ρ-equation, subject to the condition that $u(\rho = 0, \phi)$ is bounded. Write down the full solution to the axisymmetric problem in terms of $P_n(\cos\phi)$.

5. Consider the boundary condition

$$u(\rho = R, \phi) = U \cos\phi \,,$$

where U is a constant. Find the solution for $u(\rho, \phi)$ which satisfies this boundary condition at $\rho = R$.

2 Green's functions I. Consider the ordinary differential equation,

$$y'' - 4y = f(x)\,, \qquad y'(0) = 0 \qquad \text{and} \qquad y(+\infty) \to 0\,.$$

Answer the following:

(a) Find the Green's function $G(x, x_0)$ for this equation, which solves the problem

$$G'' - 4G = \delta(x - x_0)\,.$$

(b) Use the Green's function to find $y(x)$ corresponding to $f(x) = x$.

(c) Could you have obtained the solution without using Green's functions? [*Hint: What solution would you guess if you were using the Method of Undetermined Coefficients?*]

3 Green's Functions II. Consider the following equidimensional equation:

$$x^2 y'' + xy' - 9y = x\,,$$

with boundary conditions $y(0)$ bounded and $y'(1) = 0$.

(a) The Green's function for this equation satisfies

$$x^2 G'' + xG' - 9G = \delta(x - x_0)\,,$$

along with the same boundaries as $y(x)$. Solve for the Green's function. [*The most efficient way to find the jump condition for the Green's function is to express the equation in self-adjoint (i.e. Sturm–Liouville) form).*]

(b) Use the Green's function to solve for for $y(x)$.

4 Variation of parameters. Consider the following inhomogeneous version of Bessel's equation:

$$x^2 \frac{\mathrm{d}^2 u}{\mathrm{d}x^2} + x\frac{\mathrm{d}u}{\mathrm{d}x} + \left(x^2 - n^2\right) u = f(x)\,,$$

where n is an integer. Two linearly independent solutions to the homogeneous problem (the problem with $f(x) = 0$) are

$$u_1(x) = J_n(x) \qquad \text{and} \qquad u_2(x) = Y_n(x)\,.$$

Answer the following:

1. Put Bessel's equation into the Sturm-Liouville form, and identify $p(x)$.

2. We showed in class pW is equal to a constant. With the choice of u_1 and u_2 given above, the constant is $c = 2\pi^{-1}$. Using this fact along with the boundary conditions

$$u(1) = 0\,, \qquad \text{and} \qquad u(2) = 0\,,$$

write down the "variation of parameters solutions" $u = v_1 u_2 + v_2 u_2$ by solving for $v_1(x)$ and $v_2(x)$ You may leave $v_1(x)$ and $v_2(x)$ in terms of unevaluated integrals.