Homework 5

Due May 20, 2015.

1 Legendre polynomials on a tidally locked planet. The steady heat distribution in a solid sphere is governed by Laplace's equation in spherical coordinates,

$$\nabla^2 u = \frac{1}{\rho^2}\frac{\partial}{\partial\rho}\left(\rho^2\frac{\partial u}{\partial\rho}\right) + \frac{1}{\rho^2\sin\phi}\frac{\partial}{\partial\phi}\left(\sin\phi\frac{\partial u}{\partial\phi}\right) + \frac{1}{\rho^2\sin^2\phi}\frac{\partial^2 u}{\partial\theta^2} = 0\,.$$

In this equation, ρ is the radial coordinate extending from $\rho = 0$ at the center of the planet to $\rho = R$ at the surface. The coordinate ϕ is called colatitude and lies in the range $(0, \pi)$, with 0 being at the North Pole and π at the South Pole. θ is the longitudinal angle, which goes from 0 to 2π. Consider the following:

1. Separate variables by writing $u = S(\rho, \phi)q(\theta)$. Solve the equation for θ.

2. Now assume that u does not depend on θ. This assumption means that we are looking for "axisymmetric" solutions. Separate variables again by writing $S(\rho, \phi) = f(\rho)g(\phi)$ and show that $g(\phi)$ satisfies

$$\frac{\mathrm{d}}{\mathrm{d}\phi}\left(\sin\phi\frac{\mathrm{d}g}{\mathrm{d}\phi}\right) + \mu\sin\phi\, g = 0\,,$$

   where μ is an eigenvalue. The solutions to this Sturm-Liouville equation are called Legendre polynomials. There are two solutions: only one of them is bounded at both $\phi = 0$ and $\phi = \pi$, which happens for the eigenvalue $\mu = n(n+1)$, where $n = 0, 1, 2...$ is an integer. Denote this solution $g_n(\phi) = P_n(\cos\phi)$, where P_n is a polynomial. The first three P_n are

$$P_0 = 1\,,$$
$$P_1 = \cos\phi\,,$$
$$P_2 = \tfrac{1}{2}\left(3\cos^2\phi - 1\right) = \tfrac{1}{4}\left(3\cos 2\phi + 1\right)\,.$$

3. Observing that the Legendre equation is a Sturm-Liouville eigenvalue problem, write down the orthogonality relation satisfied by the Legendre polynomials. [*Note: it is not necessary to derive the orthogonality relation from scratch; instead simply write down the results of Sturm-Liouville theory as they apply to this particular problem.*]

4. Solve the ρ-equation, subject to the condition that $u(\rho = 0, \phi)$ is bounded. Write down the full solution to the axisymmetric problem in terms of $P_n(\cos\phi)$.

5. Consider the boundary condition

$$u(\rho = R, \phi) = U \cos\phi \,,$$

where U is a constant. Find the solution for $u(\rho, \phi)$ which satisfies this boundary condition at $\rho = R$.

Solution.

0. *Note.* This problem is for a fully solid sphere, such that we will have a boundary condition only at the surface $\rho = R$. For θ, we only have to impose that u is periodic in θ, since the points $\theta = 0$ and $\theta = 2\pi$ are *the same.* The points at ϕ at $\phi = 0$ and $\phi = \pi$ are also not actual boundaries in the sphere, but rather correspond to the north and south pole. We simply specify that u is bounded there, just like we specify that u is bounded at $\rho = 0$.

1. We propose $u = S(\rho, \phi)q(\theta)$ and plug this into $\nabla^2 u$. We find

$$0 = \nabla^2 u = \frac{q}{\rho^2}\frac{\partial}{\partial\rho}\left(\rho^2\frac{\partial S}{\partial\rho}\right) + \frac{q}{\rho^2\sin\phi}\frac{\partial}{\partial\phi}\left(\sin\phi\frac{\partial S}{\partial\phi}\right) + \frac{S}{\rho^2\sin^2\phi}\frac{\mathrm{d}^2 q}{\mathrm{d}\theta^2}\,.$$

We then isolate the part of the equation that depends on θ by multiplying by $\rho^2\sin^2\phi/Sq$ and moving the θ-dependent part to the other side of the equals sign. In the ordinary way in separation of variables, we find the two parts of the equation dependent either on (ρ, ϕ) or just θ can be equal to each other only if they are equal to a constant. In other words, we find

$$\frac{\sin^2\phi}{S}\frac{\partial}{\partial\rho}\left(\rho^2\frac{\partial S}{\partial\rho}\right) + \frac{\sin\phi}{S}\frac{\partial}{\partial\phi}\left(\sin\phi\frac{\partial S}{\partial\phi}\right) \quad = \quad -\frac{1}{q}\frac{\mathrm{d}^2 q}{\mathrm{d}\theta^2} \quad = \quad \lambda\,.$$

In the above we have defined the separation constant λ. The θ-equation is

$$\frac{\mathrm{d}^2 q}{\mathrm{d}\theta^2} + \lambda q = 0\,, \qquad \text{with } q \text{ periodic, so } q(0) = q(2\pi)\,.$$

The solution is

$$q = a_m\cos(m\theta) + b_m\sin(m\theta)\,,$$

where m is an integer and $\lambda = m^2$.

2. If u does not depend on θ, Laplace's equation reduces to

$$0 = \frac{1}{\rho^2}\frac{\partial}{\partial\rho}\left(\rho^2\frac{\partial u}{\partial\rho}\right) + \frac{1}{\rho^2\sin\phi}\frac{\partial}{\partial\phi}\left(\sin\phi\frac{\partial u}{\partial\phi}\right)\,.$$

This is the "axisymmetric" form of Laplace's equation. We propose that $u(\rho, \phi) = f(\rho)g(\phi)$, so that

$$\frac{g}{\rho^2}\frac{\mathrm{d}}{\mathrm{d}\rho}\left(\rho^2\frac{\mathrm{d}f}{\mathrm{d}\rho}\right) + \frac{1}{\rho^2\sin\phi}\frac{\mathrm{d}}{\mathrm{d}\phi}\left(\sin\phi\frac{\mathrm{d}g}{\mathrm{d}\phi}\right) = 0\,.$$

Multiplying by ρ^2/fg and moving the g-part to the other side of the equals sign, we find

$$\frac{1}{f}\frac{\mathrm{d}}{\mathrm{d}\rho}\left(\rho^2\frac{\mathrm{d}f}{\mathrm{d}\rho}\right) = -\frac{1}{g\sin\phi}\frac{\mathrm{d}}{\mathrm{d}\phi}\left(\sin\phi\frac{\mathrm{d}g}{\mathrm{d}\phi}\right) = \mu\,.$$

where we have defined a separation parameter μ. The ϕ-equation is therefore

$$\frac{\mathrm{d}}{\mathrm{d}\phi}\left(\sin\phi\frac{\mathrm{d}g}{\mathrm{d}\phi}\right) + \mu\sin\phi g = 0\,.$$

As noted in the problem, the solutions to this equation are called "Legendre polynomials" and the standard notation for them is $g_n = P_n(\cos\phi)$. An amazing property of the Legendre polynomials is that the eigenvalue associated with P_n is $\mu = n(n+1)$, where n is an integer.

3. The Sturm-Liouville form is

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(p(x)\frac{\mathrm{d}w}{\mathrm{d}x}\right) + qw + \lambda\sigma w = 0\,.$$

Thus in the ϕ-equation, we have $p = \sin\phi$, $q = 0$, $\sigma = \sin\phi$, and $\lambda = \mu$. The orthogonality relation is

$$\int_0^\pi P_n(\cos(\phi))P_\ell(\cos\phi)\sin\phi\,\mathrm{d}\phi = 0\,.$$

Note that this orthogonality relation for Legendre polynomials is often given in terms of the coordinate $x = \cos\phi$. You can convince yourself this is true by making the substitution $x = \cos\phi$, which implies that $\mathrm{d}x = -\sin\phi\,\mathrm{d}\phi$ with $x \in (1,-1)$.

4. The ρ-equation is

$$\frac{\mathrm{d}}{\mathrm{d}\rho}\left(\rho^2\frac{\mathrm{d}f}{\mathrm{d}\rho}\right) - \mu f = 0\,.$$

Recall that $\mu = n(n+1)$, where n is an integer. Therefore this equation becomes

$$\rho^2\frac{\mathrm{d}^2 f}{\mathrm{d}\rho^2} + 2\rho\frac{\mathrm{d}f}{\mathrm{d}\rho} - n(n+1)f = 0\,.$$

We can solve this by looking for solutions with $f \sim \rho^\alpha$. This yields a characteristic equation for α,

$$\alpha^2 + \alpha - n(n+1) = 0\,.$$

If you notice that $\alpha^2 + \alpha = \alpha(\alpha+1)$, you can think about the symmetry of the equation and guess that two solutions are $\alpha = n$ and $\alpha = -n-1$. An alternative method is the quadratic equation, which yields

$$\begin{aligned}
\alpha_\pm &= \tfrac{1}{2}\left(-1 \pm \sqrt{1+4n^2+4n}\right)\,, &(1)\\
&= \tfrac{1}{2}\left(-1 \pm \sqrt{(2n+1)^2}\right)\,, &(2)\\
&= \tfrac{1}{2}\left(-1 \pm (2n+1)\right)\,, &(3)\\
&= n \qquad \text{and} \qquad -n-1\,. &(4)
\end{aligned}$$

The solution for f is therefore

$$f = c_n \rho^n + d_n \rho^{-n-1}.$$

Notice that the solution $d_n \rho^{-n-1}$ goes to infinity as $\rho \to 0$, which means that it cannot be part of our solution. The only solution is therefore $f = c_n \rho^n$. The full solution is therefore

$$u = \sum_{n=0}^{\infty} c_n \rho^n P_n(\cos\phi).$$

5. We consider the boundary condition $u(\rho = R, \phi) = U\cos\phi$. Notice that $P_1(\cos\phi) = \cos\phi$. Therefore when we apply the boundary condition, we may write

$$U\cos\phi = UP_1(\cos\phi) = \sum_{n=1}^{\infty} c_n R^n P_n(\cos\phi).$$

At this point one can legitimately and validly simply guess that $c_1 = U/R$, since $u = U(\rho/R)\cos\phi$ certainly solves the governing equation and satisfies the boundary condition. If one is interested in a more systematic (and longer) method they can multiply the equation by $P_k(\cos\phi)\sin\phi$ and integrate from $\phi = 0$ to $\phi = \pi$. Using the orthogonality condition, this yields

$$\int_0^{\pi} UP_1(\cos\phi)P_k(\cos\phi)\sin\phi\,\mathrm{d}\phi = c_k R^k \int_0^{\pi} P_k^2(\cos\phi)\sin\phi\,\mathrm{d}\phi.$$

We can then notice that according to the orthogonality of P_k, the integral on the left is only non-zero when $k = 1$. Plugging in $k = 1$ then yields

$$U = c_1 R, \qquad \text{or} \qquad c_1 = \frac{U}{R}.$$

The total solution is therefore

$$u(\rho, \phi) = U\frac{\rho}{R}\cos\phi.$$

2 Green's functions I. Consider the ordinary differential equation,

$$y'' - 4y = f(x), \qquad y'(0) = 0 \qquad \text{and} \qquad y(+\infty) \to 0.$$

Answer the following:

(a) Find the Green's function $G(x, x_0)$ for this equation, which solves the problem

$$G'' - 4G = \delta(x - x_0).$$

(b) Use the Green's function to find $y(x)$ corresponding to $f(x) = x$.

(c) Could you have obtained the solution without using Green's functions? [*Hint: What solution would you guess if you were using the Method of Undetermined Coefficients?*]

Solution.

1. The Green's function solves

$$G'' - 4G = \delta(x - x_0)\,, \qquad \text{with} \qquad G'(0) = 0 \quad \text{and} \quad G(+\infty) \to 0\,.$$

The jump condition is found by integrating the equation from $x = x_0 - \epsilon$ to $x = x_0 + \epsilon$ and taking the limit as $\epsilon \to 0$. The delta function is *defined* such that

$$\lim_{\epsilon \to 0} \int_{x_0-\epsilon}^{x_0+\epsilon} \delta(x - x_0)\, \mathrm{d}x = 1\,.$$

The integral over G vanishes because G is bounded and the interval of integration shrinks to zero. However, the integral over G'' yields a jump representing the jump in G between $G'(x_0^-)$ and $G'(x_0^+)$,

$$G'(x_0^+) - G'(x_0^-) = 1\,.$$

We also have that G is continuous at x_0 (this is required so that G' exists), such that $G(x_0^+) = G(x_0^-)$. A good name for these "patching" conditions is the jump and continuity condition, respectively.

The solution to the G-equation is

$$G = \begin{cases} A\mathrm{e}^{2x} + B\mathrm{e}^{-2x} & \text{for} \quad x < x_0 \\ C\mathrm{e}^{2x} + D\mathrm{e}^{-2x} & \text{for} \quad x > x_0 \end{cases}$$

The boundary condition $G'(0) = 0$ implies that $A = B$. The boundary condition $G(+\infty) \to 0$ implies that $C = 0$. The Green's function becomes

$$G = \begin{cases} A\left(\mathrm{e}^{2x} + \mathrm{e}^{-2x}\right) & \text{for} \quad x < x_0 \\ D\mathrm{e}^{-2x} & \text{for} \quad x > x_0 \end{cases}$$

The continuity condition implies

$$A\left(\mathrm{e}^{2x_0} + \mathrm{e}^{-2x_0}\right) = D\mathrm{e}^{-2x_0} \qquad \Longrightarrow \qquad D = A\left(\mathrm{e}^{4x_0} + 1\right)\,.$$

Then G and G' are

$$G = \begin{cases} A\left(\mathrm{e}^{2x} + \mathrm{e}^{-2x}\right) & \text{for} \quad x < x_0 \\ A\left(\mathrm{e}^{4x_0} + 1\right)\mathrm{e}^{-2x} & \text{for} \quad x > x_0 \end{cases}$$

and

$$G' = \begin{cases} 2A\left(\mathrm{e}^{2x} - \mathrm{e}^{-2x}\right) & \text{for} \quad x < x_0 \\ -2A\left(\mathrm{e}^{4x_0} + 1\right)\mathrm{e}^{-2x} & \text{for} \quad x > x_0 \end{cases}$$

The jump condition them implies that

$$1 = 2A\Bigg(-\mathrm{e}^{-2x_0} - \mathrm{e}^{2x_0} + \mathrm{e}^{-2x_0} - \mathrm{e}^{2x_0}\Bigg)\,, \tag{5}$$

$$= -4A\mathrm{e}^{2x_0} \tag{6}$$

which implies that $A = -\frac{1}{4}\mathrm{e}^{-2x_0}$, and that the Green's function is

$$G = \begin{cases} -\frac{1}{4}\mathrm{e}^{-2x_0}\left(\mathrm{e}^{-2x} + \mathrm{e}^{2x}\right) & \text{for} \quad x < x_0 \\ -\frac{1}{4}\left(\mathrm{e}^{-2x_0} + \mathrm{e}^{2x_0}\right)\mathrm{e}^{-2x} & \text{for} \quad x > x_0 \end{cases}$$

The Green's function has the property that $G(x, x_0) = G(x_0, x)$. This means that the response at x to a point source at x_0 is equivalent to the response at x_0 to a point source at x. This is physical symmetry implied by the definition of the problem, and is not true in general. It is also difficult to deduce unless you explicitly calculate the Green's function.

2. With the Green's function in hand, we now know the general particular solution to the problem

$$y'' - 4y = f(x), \qquad y'(0) = 0 \qquad \text{and} \qquad y(+\infty) \to 0.$$

is

$$y(x) = \int_0^\infty f(x_0)G(x, x_0)\,\mathrm{d}x_0, \tag{7}$$

$$= \underbrace{\int_0^x f(x_0)G(\ x > x_0\ , x_0)\,\mathrm{d}x_0}_{\text{use } G \text{ for } x > x_0} + \underbrace{\int_x^\infty f(x_0)G(\ x < x_0\ , x_0)\,\mathrm{d}x_0}_{\text{use } G \text{ for } x < x_0}, \tag{8}$$

$$= -\tfrac{1}{4}\mathrm{e}^{-2x}\int_0^x f(x_0)\left(\mathrm{e}^{-2x_0} + \mathrm{e}^{2x_0}\right)\,\mathrm{d}x_0 - \tfrac{1}{4}\left(\mathrm{e}^{-2x} + \mathrm{e}^{2x}\right)\int_x^\infty f(x_0)\mathrm{e}^{-2x_0}\,\mathrm{d}x_0. \tag{9}$$

Now we plug in $f(x_0) = x_0$. Let's do they easy integral first. Using integration by parts, we find

$$\int_x^\infty x_0\mathrm{e}^{-2x_0}\,\mathrm{d}x_0 = -\tfrac{1}{2}x_0\,\mathrm{e}^{-2x_0}\Big|_x^\infty + \tfrac{1}{2}\int_x^\infty \mathrm{e}^{-2x_0}\,\mathrm{d}x_0,, \tag{10}$$

$$= \tfrac{1}{2}x\,\mathrm{e}^{-2x} + \tfrac{1}{4}\mathrm{e}^{-2x}, \tag{11}$$

$$= \tfrac{1}{2}(x + \tfrac{1}{2})\mathrm{e}^{-2x}. \tag{12}$$

The second term is therefore

$$-\tfrac{1}{4}\left(\mathrm{e}^{-2x} + \mathrm{e}^{2x}\right)\int_x^\infty x_0\mathrm{e}^{-2x_0}\,\mathrm{d}x_0 = -\tfrac{1}{8}(x + \tfrac{1}{2})\left(\mathrm{e}^{-4x} + 1\right), \tag{13}$$

$$= -\tfrac{1}{8}x\mathrm{e}^{-4x} - \tfrac{1}{16}\mathrm{e}^{-4x} - \tfrac{1}{8}x - \tfrac{1}{16}. \tag{14}$$

Notice that one of the terms in the first integral is very similar to what we just calculated; therefore we easily obtain

$$\int_0^x x_0\mathrm{e}^{-2x_0}\,\mathrm{d}x_0 = -\tfrac{1}{2}x_0\,\mathrm{e}^{-2x_0}\Big|_0^x + \tfrac{1}{2}\int_0^x \mathrm{e}^{-2x_0}\,\mathrm{d}x_0, \tag{15}$$

$$= -\tfrac{1}{2}x\,\mathrm{e}^{-2x} + \tfrac{1}{4}\left(1 - \mathrm{e}^{-2x}\right), \tag{16}$$

$$= -\tfrac{1}{2}x\mathrm{e}^{-2x} - \tfrac{1}{4}\mathrm{e}^{-2x} + \tfrac{1}{4}. \tag{17}$$

This means that

$$-\tfrac{1}{4}\mathrm{e}^{-2x}\int_0^x x_0\mathrm{e}^{-2x_0}\,\mathrm{d}x_0 = \tfrac{1}{8}x\mathrm{e}^{-4x} + \tfrac{1}{16}\mathrm{e}^{-4x} + \tfrac{1}{16}\mathrm{e}^{-2x}\,.$$

There's only one more integral to calculate, which is

$$\int_0^x x_0\mathrm{e}^{2x_0}\,\mathrm{d}x_0 = \tfrac{1}{2}x_0\mathrm{e}^{2x_0}\Big|_0^x - \tfrac{1}{2}\int_0^x \mathrm{e}^{2x_0}\,\mathrm{d}x_0\,, \tag{18}$$

$$= \tfrac{1}{2}x\,\mathrm{e}^{2x} - \tfrac{1}{4}\left(\mathrm{e}^{2x} - 1\right)\,. \tag{19}$$

Therefore,

$$-\tfrac{1}{4}\mathrm{e}^{-2x}\int_0^x x_0\mathrm{e}^{2x_0}\,\mathrm{d}x_0 = -\tfrac{1}{8}x - \tfrac{1}{16}\mathrm{e}^{-2x} + \tfrac{1}{16}\,.$$

Notice that the parts dependent on e^{-4x} all cancel out. Adding the remaining components, we find

$$y(x) = -\tfrac{1}{8}x + \tfrac{1}{16} - \tfrac{1}{16}\mathrm{e}^{-2x} - \tfrac{1}{8}x - \tfrac{1}{16}\mathrm{e}^{-2x} + \tfrac{1}{16} \tag{20}$$

$$= -\tfrac{1}{4}x - \tfrac{1}{8}\mathrm{e}^{-2x}\,. \tag{21}$$

And that's the solution.

3. We could have guessed the solution by observing that the particular problem has a straightforward solution, namely

$$y_p = -\tfrac{1}{4}x\,.$$

To complete the problem we then add the the homogeneous solution which satisfies the boundary condition at $+\infty$, e^{-2x}. Our guess is then

$$y = -\tfrac{1}{4}x + c\mathrm{e}^{-2x}\,.$$

Then, taking a derivative and setting $y'(0) = 0$ implies that $c = -1/8$.

3 Green's Functions II. Consider the following equidimensional equation:

$$x^2y'' + xy' - 9y = x\,,$$

with boundary conditions $y(0)$ bounded and $y'(1) = 0$.

(a) The Green's function for this equation satisfies

$$x^2G'' + xG' - 9G = \delta(x - x_0)\,,$$

along with the same boundaries as $y(x)$. Solve for the Green's function. [*The most efficient way to find the jump condition for the Green's function is to express the equation in self-adjoint (i.e. Sturm–Liouville) form).*]

(b) Use the Green's function to solve for for $y(x)$.

Solution.

1. To find the Green's function, we integrate the Green's function equation over a vanishing neighborhood containing x_0: in other words, we integrate from $x = x_0 - \epsilon$ to $x = x_0 + \epsilon$ and take the limit as $\epsilon \to 0$. As in problem 2, the delta function integrates to 1. Also, the terms involving G and xG' are bounded, and therefore vanish when $\epsilon \to 0$. To evaluate the integral over x^2G'', we use integration by parts, which implies

$$\lim_{\epsilon\to 0}\int_{x_0-\epsilon}^{x_0+\epsilon} x^2G''\,\mathrm{d}x = \lim_{\epsilon\to 0}\int_{x_0-\epsilon}^{x_0+\epsilon} \frac{\mathrm{d}}{\mathrm{d}x}\left(x^2G'\right) - 2xG'\,\mathrm{d}x\,, \tag{22}$$

$$= \lim_{\epsilon\to 0}\left[x^2G'\Big|_{x_0-\epsilon}^{x_0+\epsilon} - \int_{x_0-\epsilon}^{x_0+\epsilon} 2xG'\,\mathrm{d}x\right], \tag{23}$$

$$= x_0^2\Big[G'(x_0^+) - G'(x_0^-)\Big]\,. \tag{24}$$

In the final step, we have used the fact that $2xG'$ is bounded and therefore goes to zero when integrated over a vanishingly small interval. The only other non-zero term from the equation is the "1" associated with the delta function; and the jump condition is therefore

$$G'(x_0^+) - G'(x_0^-) = x_0^{-2}\,.$$

In addition to the jump condition we have the "continuity" condition, which requires that

$$G(x_0^+) = G(x_0^-)\,.$$

We need this condition in order for G' to be bounded. Now, the Green's function equation is

$$x^2G'' + xG' - 9G = \delta(x - x_0)\,, \qquad \text{with} \qquad |G(0)| < \infty \qquad \text{and} \qquad G'(1) = 0\,.$$

We solve this by setting the right side equal to zero, finding the form for G for both $x < x_0$ and $x > x_0$, and then applying the boundary conditions. Since this is an equidimensional equation, we solve it by guessing a solution of the form $G = x^\alpha$. This yields an equation for α:

$$\alpha^2 - 9 = 0\,,$$

which implies that $\alpha = \pm 3$. The general solution is therefore

$$G = \begin{cases} Ax^3 + Bx^{-3} & \text{for} \quad x < x_0 \\ Cx^3 + Dx^{-3} & \text{for} \quad x > x_0 \end{cases}$$

The condition at $x = 0$ implies that $B = 0$. The condition at $x = 1$ implies that $C - D = 0$. We therefore have

$$G = \begin{cases} Ax^3 & \text{for} \quad x < x_0 \\ C\left(x^3 + x^{-3}\right) & \text{for} \quad x > x_0 \end{cases}$$

The continuity condition requires that $Ax_0^3 = C(x_0^3 + x_0^{-3})$, which implies that

$$A = C\left(1 + x_0^{-6}\right) .$$

There we have

$$G = \begin{cases} C\left(1 + x_0^{-6}\right)x^3 & \text{for} \quad x < x_0 \\ C\left(x^3 + x^{-3}\right) & \text{for} \quad x > x_0 \end{cases}$$

and

$$G' = \begin{cases} 3C\left(1 + x_0^{-6}\right)x^2 & \text{for} \quad x < x_0 \\ 3C\left(x^2 - x^{-4}\right) & \text{for} \quad x > x_0 \end{cases}$$

The jump condition therefore implies that

$$x_0^{-2} = 3C\left(x_0^2 - x_0^{-4} - \left[x_0^2 + x_0^{-4}\right]\right), \tag{25}$$

$$= -6Cx_0^{-4}, \tag{26}$$

and therefore $C = -\frac{1}{6}x_0^2$. So the Green's function is,

$$G = \begin{cases} -\frac{1}{6}\left(x_0^2 + x_0^{-4}\right)x^3 & \text{for} \quad x < x_0 \\ -\frac{1}{6}x_0^2\left(x^3 + x^{-3}\right) & \text{for} \quad x > x_0 \end{cases}$$

2. With the Green's function, we know that the general solution to the problem

$$x^2y'' + xy' - 9y = f(x), \qquad \text{with} \qquad |y(0)| < \infty \qquad \text{and} \qquad y'(1) = 0,$$

is

$$y(x) = \int_0^1 f(x_0)G(x, x_0)\,\mathrm{d}x_0, \tag{27}$$

$$= \underbrace{\int_0^x f(x_0)G(x > x_0, x_0)\,\mathrm{d}x_0}_{\text{use } G \text{ for } x > x_0} + \underbrace{\int_x^1 f(x_0)G(x < x_0, x_0)\,\mathrm{d}x_0}_{\text{use } G \text{ for } x < x_0}, \tag{28}$$

$$= -\frac{1}{6}\left(x^3 + x^{-3}\right)\int_0^x x_0^2 f(x_0)\,\mathrm{d}x_0 - \frac{1}{6}x^3\int_x^1\left(x_0^2 + x_0^{-4}\right)f(x_0)\,\mathrm{d}x_0. \tag{29}$$

With $f(x) = x$ the integrals are pretty easy. The first is

$$\int_0^x x_0^3\,\mathrm{d}x = \tfrac{1}{4}x^4,$$

and the second is

$$\int_x^1 x_0^3 + x_0^{-3}\,\mathrm{d}x = \tfrac{1}{4}x_0^4 - \tfrac{1}{2}x_0^{-2}\Big|_x^1, \tag{30}$$

$$= -\tfrac{1}{4} - \tfrac{1}{4}x^4 + \tfrac{1}{2}x^{-2}. \tag{31}$$

Putting the pieces together, we find

$$y(x) = -\tfrac{1}{6}\left(x^3 + x^{-3}\right)\left(\tfrac{1}{4}x^4\right) - \tfrac{1}{6}x^3\left(-\tfrac{1}{4} - \tfrac{1}{4}x^4 + \tfrac{1}{2}x^{-2}\right), \tag{32}$$

$$= -\tfrac{1}{24}\left(x^7 + x - x^3 - x^7 + 2x\right), \tag{33}$$

$$= -\tfrac{1}{8}x + \tfrac{1}{24}x^3. \tag{34}$$

4 Variation of parameters. Consider the following inhomogeneous version of Bessel's equation:

$$x^2\frac{\mathrm{d}^2u}{\mathrm{d}x^2} + x\frac{\mathrm{d}u}{\mathrm{d}x} + \left(x^2 - n^2\right)u = f(x),$$

where n is an integer. Two linearly independent solutions to the homogeneous problem (the problem with $f(x) = 0$) are

$$u_1(x) = J_n(x) \qquad \text{and} \qquad u_2(x) = Y_n(x).$$

Answer the following:

1. Put Bessel's equation into the Sturm-Liouville form, and identify $p(x)$.
2. We showed in class pW is equal to a constant. With the choice of u_1 and u_2 given above, the constant is $c = 2\pi^{-1}$. Using this fact along with the boundary conditions

   $$u(1) = 0, \qquad \text{and} \qquad u(2) = 0,$$

   write down the "variation of parameters solutions" $u = v_1u_1 + v_2u_2$ by solving for $v_1(x)$ and $v_2(x)$. You may leave $v_1(x)$ and $v_2(x)$ in terms of unevaluated integrals.

Solution.

1. To put Bessel's equation in Strum-Liouville form, we note that

   $$x\frac{\mathrm{d}}{\mathrm{d}x}\left(x\frac{\mathrm{d}u}{\mathrm{d}x}\right) = x^2\frac{\mathrm{d}^2u}{\mathrm{d}x^2} + x\frac{\mathrm{d}u}{\mathrm{d}x},$$

   and so we can use this fact to write

   $$x\frac{\mathrm{d}}{\mathrm{d}x}\left(x\frac{\mathrm{d}u}{\mathrm{d}x}\right) + \left(x^2 - n^2\right)u = f(x).$$

   Then, dividing by x yields the Sturm-Liouville form:

   $$\frac{\mathrm{d}}{\mathrm{d}x}\left(x\frac{\mathrm{d}u}{\mathrm{d}x}\right) + \left(x - \frac{n^2}{x}\right)u = \frac{f(x)}{x}.$$

   We therefore have that $p = x$, and $q = x - n^2/x$. The forcing term if $F(x) = f(x)/x$.

2. Given the two solutions u_1, and u_2, we showed in class that if we define the "Wronskian" according to

$$W = u_1 \frac{\mathrm{d}u_2}{\mathrm{d}x} - u_2 \frac{\mathrm{d}u_1}{\mathrm{d}x},$$

then we have the following remarkable fact if u_1 and u_2 are homogeneous solutions for the Sturm-Liouville operator:

$$W = \frac{c}{p},$$

where $p(x)$ is the function appearing in the Strum-Liouville operator and c is a constant. In fact, c is actually defined by $p(x)W(x)$. It is not at all obvious what c should be in advance for an arbitrary problem. In this case, we have that $c = 2\pi^{-1}$. But c will take other values for other choices of u_1 and u_2, or for other Sturm-Liouville operators.

The amazing thing about having c is that this means we can calculate

$$\frac{\mathrm{d}v_1}{\mathrm{d}x} = -\frac{Fu_2}{pW} = -\frac{Fu_2}{c},$$

and

$$\frac{\mathrm{d}v_2}{\mathrm{d}x} = \frac{Fu_1}{pW} = \frac{Fu_1}{c}.$$

We can therefore integrate these expressions. Let's choose the bounds of integration to be at $x = 1$. We can choose them to be anywhere, but arbitrary choices will lead to irritatingly large expressions for the constants that are needed to satisfy the boundary conditions. Thus we have

$$v_1(x) = -\int_1^x \frac{\pi f(z) Y_n(z)}{2z}\,\mathrm{d}z + d_1,$$

and

$$v_2(x) = \int_1^x \frac{\pi f(z) J_n(z)}{2z}\,\mathrm{d}z + d_2,$$

Where d_1 and d_2 are the constants we must determine using the boundary conditions. The solution is then

$$u = v_1 u_1 + v_1 u_1, \tag{35}$$

$$= -J_n(x)\int_1^x \frac{\pi f(z) Y_n(z)}{2z}\,\mathrm{d}z + d_1 J_n(x) + Y_n(x)\int_1^x \frac{\pi f(z) J_n(z)}{2z}\,\mathrm{d}z + d_2 Y_n(x). \tag{36}$$

The boundary condition $u(1) = 0$ implies that

$$0 = d_1 J_n(1) + d_2 Y_n(1),$$

which implies that

$$d_1 = -d_2 \frac{Y_n(1)}{J_n(1)} .$$

Our expression for u is then

$$u = -J_n(x) \int_1^x \frac{\pi f(z) Y_n(z)}{2z} \, \mathrm{d}z + Y_n(x) \int_1^x \frac{\pi f(z) J_n(z)}{2z} \, \mathrm{d}z + \frac{d_2}{J_n(1)} \Big[J_n(1) Y_n(x) - Y_n(1) J_n(x) \Big] .$$

The condition at $x = 2$ implies that

$$J_n(2) \int_1^2 \frac{\pi f(z) Y_n(z)}{2z} \, \mathrm{d}z - Y_n(2) \int_1^2 \frac{\pi f(z) J_n(z)}{2z} \, \mathrm{d}z = \frac{d_2}{J_n(1)} \Big[J_n(1) Y_n(2) - Y_n(1) J_n(2) \Big] .$$

This is a long and annoying expression for d_2, so let's just write the solution as

$$u = -J_n(x) \int_1^x \frac{\pi f(z) Y_n(z)}{2z} \, \mathrm{d}z + Y_n(x) \int_1^x \frac{\pi f(z) J_n(z)}{2z} \, \mathrm{d}z + C \frac{J_n(1) Y_n(x) - Y_n(1) Y_n(x)}{J_n(1) Y_n(2) - Y_n(1) J_n(2)} .$$

where

$$C \stackrel{\text{def}}{=} J_n(2) \int_1^2 \frac{\pi f(z) Y_n(z)}{2z} \, \mathrm{d}z - Y_n(2) \int_1^2 \frac{\pi f(z) J_n(z)}{2z} \, \mathrm{d}z .$$

The solution is massive and unwieldy, but it's the solution nevertheless.