Homework 6

Due June 3, 2015.

1 The convective heat equation. Heat conduction in the presence of a background flow is governed by the equation

$$\frac{\partial u}{\partial t} + U\frac{\partial u}{\partial x} = \kappa \frac{\partial^2 u}{\partial x^2},$$

where U is the background velocity. Consider an infinite domain with $u \to 0$ as $x \to \pm\infty$ and $u(x, 0) = f(x)$.

1. Solve this equation using the Fourier transform and express the solution in terms of an integral of $f(x)$ times an "influence function". [*Hint: use the convolution and shift theorems from class.*]
2. Solve for the initial condition $f(x) = \delta(x)$.

2 Steady heat conduction across a gap. Steady heat conduction in a two-dimensional domain is governed by Laplace's equation:

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0\,.$$

We consider a domain which runs from $-\infty < x < \infty$, but is bounded in y at 0 and L. The boundary conditions in y are

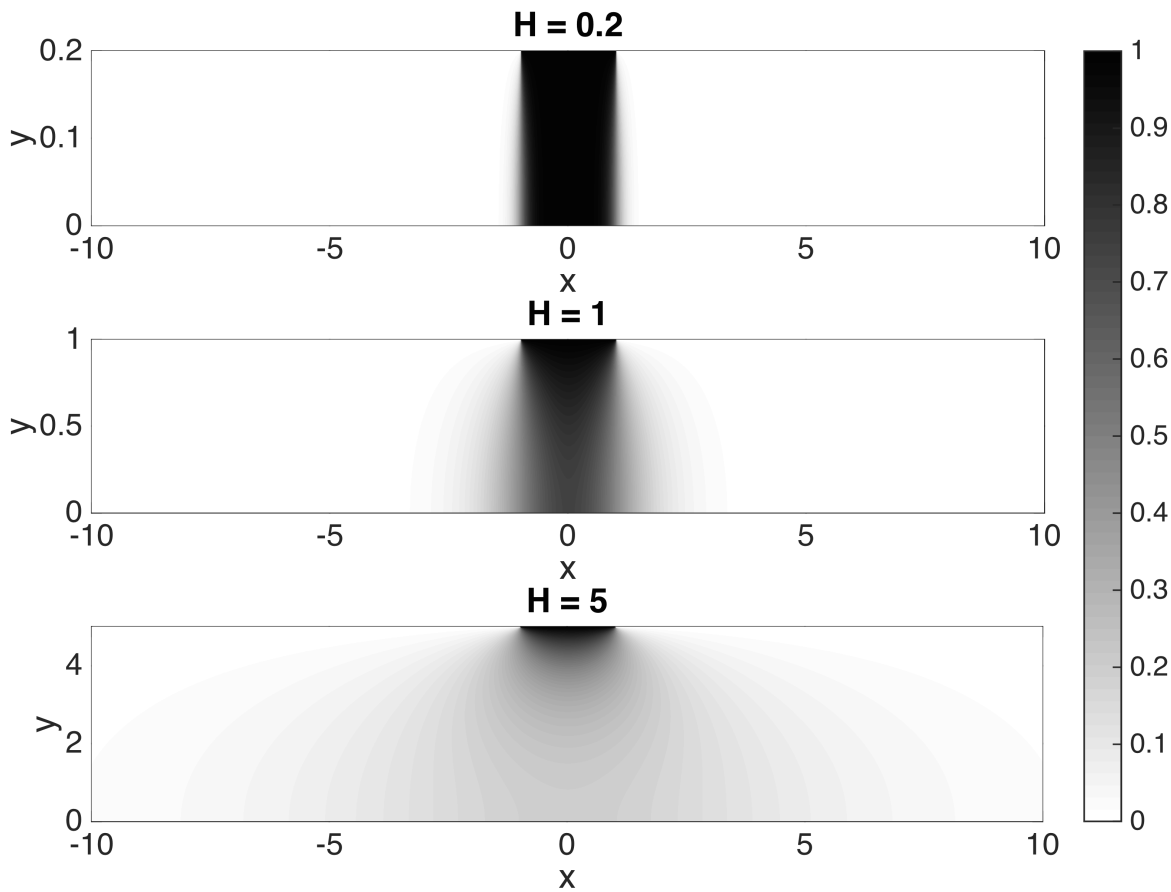
$$\frac{\partial u}{\partial y}(x, y = 0) = 0\,, \qquad \text{and} \qquad \frac{\partial u}{\partial y}(x, y = H) = f(x)\,.$$

Answer the following:

1. Use the Fourier transform of Laplace's equation in x to obtain an ODE in y for $U(\omega, y)$. Solve this equation.
2. Use the inverse Fourier transform to write the solution for $u(x, y)$ as an integral over ω.
3. Below, the solution is plotted for the boundary condition

$$f(x) = \begin{cases} 1 & \text{for} \quad -1 < x < 1\,, \\ 0 & \text{elsewhere.} \end{cases}$$

for $H = 0.2$, $H = 1$, and $H = 5$. Argue why this makes sense given the form of the integral you found for the previous question.



3 The wave equation and the Fourier transform. Consider the wave equation in an infinite domain,

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}\,,$$

with initial conditions

$$u(x,0) = f(x)\,, \qquad \text{and} \qquad \frac{\partial u}{\partial t}(x,0) = 0\,.$$

Answer the following:

(a) Write down the Fourier transform of the wave equation in terms of $U(\omega, t)$. Notice that the time-derivative transforms to

$$\frac{\partial^2 U}{\partial t^2}\,,$$

while the x-derivative term can be tackled using the derivative rule proved in problem 1.

(b) To solve the time-dependent equation you also need the Fourier transform of the initial condition; denote this $F(\omega)$. Now, solve the equation for $U(\omega, t)$ and apply the initial conditions. You should find an answer in terms of $F(\omega)$.

(c) Invert the transform to find the general solution for $u(x,t)$. *Hint: two hints will prove useful. First, recall that* $\cos(\theta)$ *can be written*

$$\cos(\theta) = \tfrac{1}{2}\left(\mathrm{e}^{-\mathrm{i}\theta} + \mathrm{e}^{\mathrm{i}\theta}\right).$$

Next, define an intermediate variable $z = x - ct$*...*

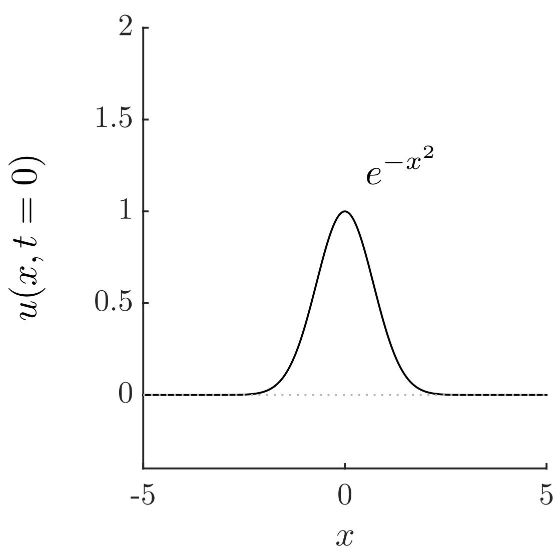
4 Method of characteristics. Consider the wave equation

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2},$$

with the initial conditions

$$u(x,0) = \mathrm{e}^{-x^2}, \qquad \text{and} \qquad \frac{\partial u}{\partial t} = 0.$$

The initial condition is given below.



Obtain the solution and sketch it at two later times.

5 The wave equation in spherical polar coordinates. Consider the wave equation

$$\frac{\partial^2 u}{\partial t^2} = c^2 \nabla^2 u,$$

in three dimensions when the solution is spherically symmetric, so that

$$\nabla^2 u = \frac{1}{\rho^2}\frac{\partial}{\partial \rho}\left(\rho^2 \frac{\partial u}{\partial \rho}\right).$$

Answer the following:

(a) Write $u(\rho, t) = \rho^{-1} w(\rho, t)$ and show that $w(\rho, t)$ obeys the one-dimensional wave equation.

(b) Hence solve for $u(\rho, t)$ in the general case.

(c) Explain why the resulting solution corresponds to one outgoing and one incoming wave.