Homework 6

Due June 3, 2015.

1 The convective heat equation. Heat conduction in the presence of a background flow is governed by the equation

$$\frac{\partial u}{\partial t} + U\frac{\partial u}{\partial x} = \kappa \frac{\partial^2 u}{\partial x^2},$$

where U is the background velocity. Consider an infinite domain with $u \to 0$ as $x \to \pm\infty$ and $u(x,0) = f(x)$.

1. Solve this equation using the Fourier transform and express the solution in terms of an integral of $f(x)$ times an "influence function". [*Hint: use the convolution and shift theorems from class.*]
2. Solve for the initial condition $f(x) = \delta(x)$.

Solution.

1. Because U is the constant background velocity in this problem, we define the Fourier transform with

   $$\hat{u}(\omega, t) = \mathcal{F}[u(x,t)] = \frac{1}{2\pi}\int_{-\infty}^{\infty} u(x,t)\mathrm{e}^{\mathrm{i}\omega x}\,\mathrm{d}x\,.$$

   Note that derivatives transform under the rule

   $$\frac{\partial}{\partial x} \to -\mathrm{i}\omega \qquad \text{or} \qquad \mathcal{F}\left[\frac{\partial u}{\partial x}\right] = -\mathrm{i}\omega\hat{u} \quad \text{and} \quad \mathcal{F}\left[\frac{\partial^2 u}{\partial x^2}\right] = -\omega^2\hat{u}\,.$$

   The Fourier transform of the convective heat equation is

   $$\frac{\partial \hat{u}}{\partial t} - \mathrm{i}\omega U\hat{u} + \kappa\omega^2\hat{u} = 0\,.$$

   This is a first-order, constant-coefficient ODE in t for $\hat{u}$,which has a solution of the form $\hat{u} = A\mathrm{e}^{kt}$. If we plug this into the equation we find $k = -\kappa\omega^2 + \mathrm{i}\omega U$, and therefore

   $$\hat{u}(\omega, t) = A(\omega)\mathrm{e}^{-\kappa\omega^2 t + \mathrm{i}\omega U t}\,.$$

   The initial condition $\hat{u}(\omega, t = 0)$ is found by taking the Fourier transform of the initial condition on u, which implies that $\hat{u}(x, t = 0) = \hat{f}(\omega)$, where $\hat{f} = \mathcal{F}[f(x)]$. Putting $t = 0$ into the above equation for $\hat{u}$ implies that $A = \hat{f}$, so that

   $$\hat{u}(\omega, t) = \hat{f}(\omega)\,\mathrm{e}^{-\kappa\omega^2 t}\mathrm{e}^{\mathrm{i}\omega U t}\,.$$

Notice that the solution for $\hat{u}$ involves multiplication by $e^{i\omega U}$. We can use the shift theorem for this part of $\hat{u}$. The shift theorem implies that if

$$\mathcal{F}^{-1}[\hat{W}(\omega)] = W(x)\,,$$

then

$$\mathcal{F}^{-1}[\hat{W}(\omega)e^{i\beta\omega}] = W(x-\beta)\,.$$

Therefore if we define $W(x)$ as

$$\hat{W}(\omega) \stackrel{\text{def}}{=} \hat{f}(\omega)\, e^{-\kappa\omega^2 t}\,,$$

We have that

$$u(x,t) = \mathcal{F}^{-1}\left[\hat{W}(\omega)e^{i\omega U t}\right] = W(x-Ut)\,.$$

The problem of finding $u(x,t)$ thus boils down to finding the inverse transform of $\hat{W}$. Notice that $\hat{W}$ involves the product of two functions: therefore, we can use the convolution theorem to evaluate $W(x)$ in physical space. The convolution theorem states that if $\hat{H}(\omega) = \hat{F}(\omega)\hat{G}(\omega)$, then $H(x)$, the inverse transform of $\hat{H}(\omega)$, is given by

$$H(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\xi)G(x-\xi)\,d\xi\,,$$

where $F(x)$ and $G(x)$ are the inverse transforms of $\hat{F}(\omega)$ and $\hat{G}(\omega)$.

Above we defined $\hat{W}(\omega)$ as $\hat{W} = \hat{f}e^{-\kappa\omega^2 t}$. So one of the functions is $\hat{f}(\omega)$; the inverse transform is known and it is just the initial condition $f(x)$. The other function is $e^{-\kappa\omega^2 t}$ – the Gaussian. The inverse transform of this function was done in class and is also given in the book. We recapitulate these results here for completeness. Define

$$\hat{g}(\omega) = e^{-(\nu\omega)^2}\,.$$

The inverse transform is then

$$g(x) = \int_{-\infty}^{\infty} e^{-(\nu\omega)^2 - i\omega x}\,d\omega\,. \tag{1}$$

To calculate this integral we "complete the square"; that is, we observe that if we define $s = \nu\omega + c$, we have

$$-s^2 = -\,(\nu\omega + c)^2 = -(\nu\omega)^2 - 2c\nu\omega - c^2\,.$$

Because we can easily calculate the integral of e^{-s^2} from $-\infty < s < \infty$, we are strongly motivated to write the integral in terms of s. Notice that if $c = ix/2\nu$ we have that $2c\nu\omega = i\omega x$, and therefore

$$-\left(\nu\omega + \frac{ix}{2\nu}\right)^2 = -(\nu\omega)^2 - i\omega x + \frac{x^2}{4\nu^2}\,,$$

or, rearranging the terms,

$$-(\nu\omega)^2 - \mathrm{i}\omega x = -\left(\nu\omega + \frac{\mathrm{i}x}{2\nu}\right)^2 - \frac{x^2}{4\nu^2}\,.$$

We can therefore put this into the inverse transform for $g(x)$ to obtain

$$g(x) = \int_{-\infty}^{\infty} \mathrm{e}^{-x^2/4\nu^2}\mathrm{e}^{-(\nu\omega+\mathrm{i}x/2\nu)^2}\,\mathrm{d}\omega\,. \tag{2}$$

We can now define s,

$$s \stackrel{\text{def}}{=} \nu\omega + \frac{\mathrm{i}x}{2\nu}\,, \qquad \Longrightarrow \qquad \frac{\mathrm{d}s}{\mathrm{d}\omega} = \nu \quad \text{and thus} \quad \mathrm{d}\omega = \nu^{-1}\mathrm{d}s\,.$$

Notice that as $\omega \to \pm\infty$, with x and ν fixed, we still have that $s \to \pm\infty$. So thankfully, the limits of the integral don't change. Further, we can pull the factor $\mathrm{e}^{-x^2/4\nu^2}$ out of integral, since it does not depend on ω. The result is that

$$g(x) = \mathrm{e}^{-x^2/4\nu^2}\int_{-\infty}^{\infty} \mathrm{e}^{-(\nu\omega+\mathrm{i}x/2\nu)^2}\,\mathrm{d}\omega\,, \tag{3}$$

$$= \mathrm{e}^{-x^2/4\nu^2}\int_{-\infty}^{\infty} \mathrm{e}^{-s^2}\left(\nu^{-1}\,\mathrm{d}s\right)\,, \tag{4}$$

$$= \frac{\sqrt{\pi}}{\nu}\mathrm{e}^{-x^2/4\nu^2}\,. \tag{5}$$

In the final step, we have used the fact that $\int_{-\infty}^{\infty} \mathrm{e}^{-s^2}\,\mathrm{d}s = \sqrt{\pi}$. This important fact should be memorized; and in the Appendix we prove it to be true.

In any case, we are now able to calculate the inverse transform of $\mathrm{e}^{-\kappa t\omega^2}$. In the above formula this corresponds to $\nu = \sqrt{\kappa t}$; thus we have

$$\mathcal{F}\left[\mathrm{e}^{-\kappa t\omega^2}\right] = \sqrt{\frac{\pi}{\kappa t}}\,\mathrm{e}^{-x^2/4\kappa t}\,.$$

Finally, if we define $G(x) = \sqrt{\frac{\pi}{\kappa t}}\,\mathrm{e}^{-x^2/4\kappa t}$, we have that

$$G(x-\xi) = \sqrt{\frac{\pi}{\kappa t}}\,\mathrm{e}^{-(x-\xi)^2/4\kappa t}\,,$$

and the convolution theorem implies that

$$W(x) = \int_{-\infty}^{\infty} f(\xi)\sqrt{\frac{\pi}{\kappa t}}\,\mathrm{e}^{-(x-\xi)^2/4\kappa t}\,\mathrm{d}\xi\,.$$

And the grand finale is that since $u(x,t) = W(x-Ut)$, we then must have that

$$u(x,t) = \frac{2}{\pi}\int_{-\infty}^{\infty} f(\xi)\sqrt{\frac{\pi}{\kappa t}}\,\mathrm{e}^{-(x-Ut-\xi)^2/4\kappa t}\,\mathrm{d}\xi\,, \tag{6}$$

$$= \frac{2}{\sqrt{\pi\kappa t}}\int_{-\infty}^{\infty} f(\xi)\,\mathrm{e}^{-(x-Ut-\xi)^2/4\kappa t}\,\mathrm{d}\xi\,. \tag{7}$$

Note that we could also write this as

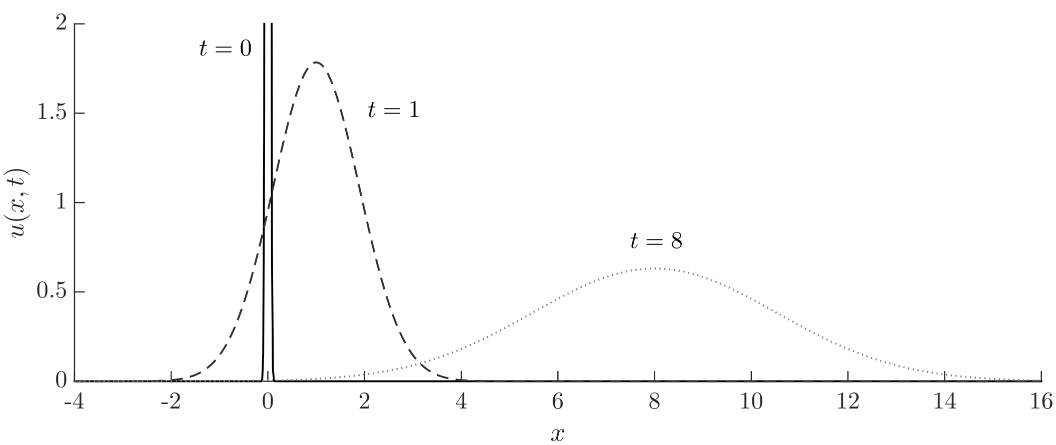
$$u(x,t) = \int_{-\infty}^{\infty} f(x - Ut - \xi)\sqrt{\frac{\pi}{\kappa t}}\, \mathrm{e}^{-\xi^2/4\kappa t}\, \mathrm{d}\xi\,.$$

2. When $f = \delta(x)$, we can easily calculate the convolution integral. We find

$$u(x,t) = \frac{2}{\sqrt{\pi\kappa t}} \int_{-\infty}^{\infty} \delta(\xi)\, \mathrm{e}^{-(x-Ut-\xi)^2/4\kappa t}\, \mathrm{d}\xi\,, \tag{8}$$

$$= \frac{2}{\sqrt{\pi\kappa t}}\, \mathrm{e}^{-(x-Ut)^2/4\kappa t}\,. \tag{9}$$

We could get the same result if we use $\hat{f}(\omega) = (2\pi)^{-1}$ and took the inverse transform of the resulting expression for $\hat{u}$. Below, the solution is plotted for $\kappa = 0.4$ and $U = 1$ at a few different times to help you understand what this looks like. Basically, our equation describes an initially concentrated pulse of heat which is both diffusing in space while being advected to the right by a steady wind.



2 Steady heat conduction across a gap. Steady heat conduction in a two-dimensional domain is governed by Laplace's equation:

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0\,.$$

We consider a domain which runs from $-\infty < x < \infty$, but is bounded in y at 0 and L. The boundary conditions in y are

$$\frac{\partial u}{\partial y}(x, y = 0) = 0\,, \qquad \text{and} \qquad \frac{\partial u}{\partial y}(x, y = H) = f(x)\,.$$

Answer the following:

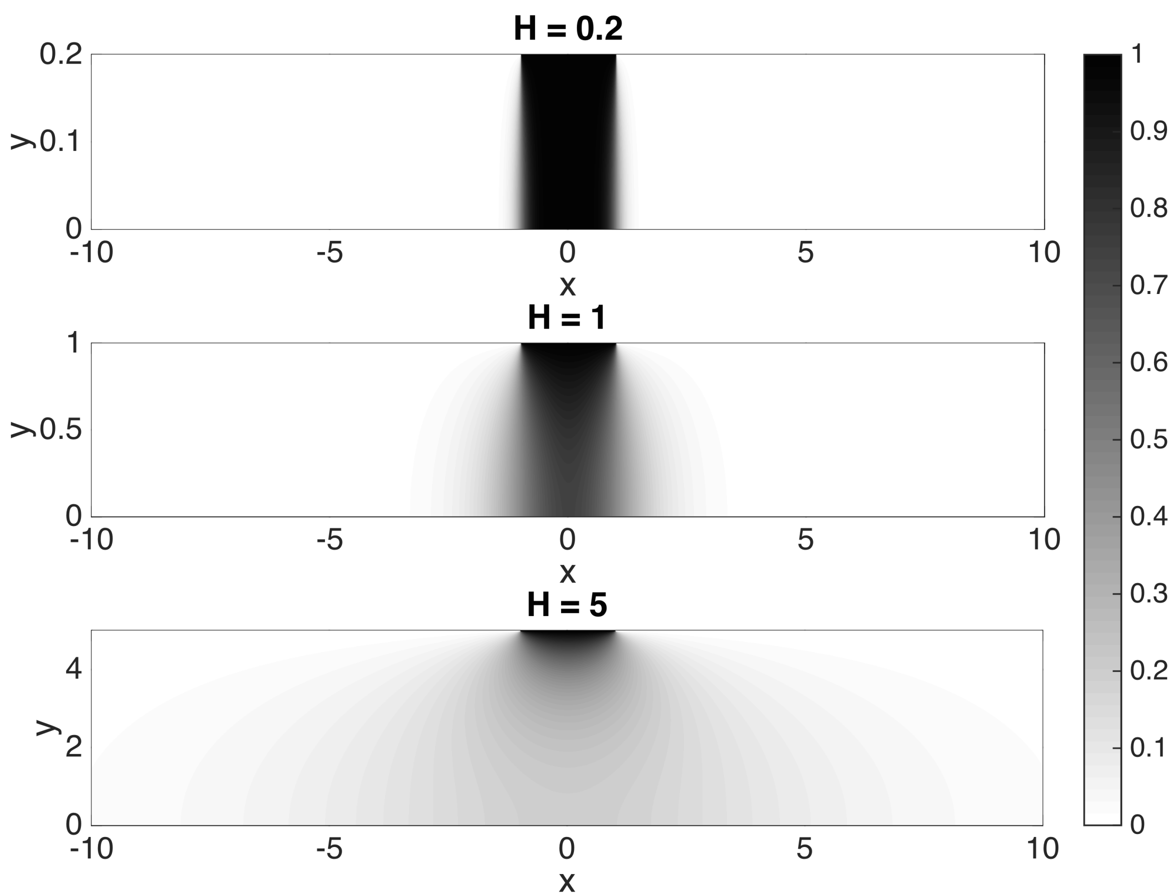
1. Use the Fourier transform of Laplace's equation in x to obtain an ODE in y for $U(\omega, y)$. Solve this equation.

2. Use the inverse Fourier transform to write the solution for $u(x, y)$ as an integral over ω.

3. Below, the solution is plotted for the boundary condition

$$f(x) = \begin{cases} 1 & \text{for} \quad -1 < x < 1\,, \\ 0 & \text{elsewhere.} \end{cases}$$

for $H = 0.2$, $H = 1$, and $H = 5$. Argue why this makes sense given the form of the integral you found for the previous question.



Solution.

1. We denote the x-Fourier transform of $u(x, y)$ with $U(\omega, y)$ and the transform of $f(x)$ with $F(\omega)$. The transform of Laplace's equation in x is then

$$\frac{\partial^2 U}{\partial y^2} - \omega^2 U = 0\,, \qquad \text{with} \qquad \frac{\partial U}{\partial y}(\omega, y = 0) = 0\,, \quad \text{and} \quad \frac{\partial U}{\partial y}(x, y = H) = F(\omega)\,.$$

The solution is exponentials. We write it in terms of $\cosh(\omega y)$ for convenience. The solution is
$$U(\omega, y) = A\cosh(\omega y) + B\sinh(\omega y)\,,$$
and the condition $\partial U/\partial y = 0$ at $y = 0$ implies that $B = 0$. Satisfying the condition at $y = H$ implies that
$$F(\omega) = A\cosh(\omega H)\,, \qquad \text{so that} \qquad A = \frac{F(\omega)}{\cosh(\omega H)}\,,$$
and
$$U(\omega, y) = F(\omega)\frac{\cosh(\omega y)}{\cosh(\omega H)}\,.$$
This is the solution for $U(\omega y)$.

2. With $U(\omega, y)$ in hand, we can find $u(x, y)$ with the inverse transform. We have
$$u(x, y) = \int_{-\infty}^{\infty} F(\omega)\frac{\cosh(\omega y)}{\cosh(\omega H)} \mathrm{e}^{-\mathrm{i}\omega x}\, \mathrm{d}\omega\,.$$

3. Observe in the above equation that when H is very large, this implies that
$$\frac{\cosh(\omega y)}{\cosh(\omega H)}$$
is extremely small except very close to $y = H$. Thus for large H – or $H = 5$ – the solution decays rapidly away from the boundary at $y = H$. On the other hand, when H is very small, we have
$$\frac{\cosh(\omega y)}{\cosh(\omega H)} \approx 1$$
over the whole domain. When that is true, the solution for $u(x, y)$ should basically just look like $f(x)$, which is exactly what we see for $H = 0.2$.

3 The wave equation and the Fourier transform. Consider the wave equation in an infinite domain,
$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}\,,$$
with initial conditions
$$u(x, 0) = f(x)\,, \qquad \text{and} \qquad \frac{\partial u}{\partial t}(x, 0) = 0\,.$$

Answer the following:

(a) Write down the Fourier transform of the wave equation in terms of $U(\omega, t)$. Notice that the time-derivative transforms to

$$\frac{\partial^2 U}{\partial t^2},$$

while the x-derivative term can be tackled using the derivative rule proved in problem 1.

(b) To solve the time-dependent equation you also need the Fourier transform of the initial condition; denote this $F(\omega)$. Now, solve the equation for $U(\omega, t)$ and apply the initial conditions. You should find an answer in terms of $F(\omega)$.

(c) Invert the transform to find the general solution for $u(x, t)$. *Hint: two hints will prove useful. First, recall that* $\cos(\theta)$ *can be written*

$$\cos(\theta) = \tfrac{1}{2}\left(\mathrm{e}^{-\mathrm{i}\theta} + \mathrm{e}^{\mathrm{i}\theta}\right).$$

Next, define an intermediate variable $z = x - ct$*...*

Solution.

1. The Fourier transform of the wave equation is

$$\frac{\partial^2 U}{\partial t^2} + c^2\omega^2 U = 0.$$

2. The Fourier transform of the initial condition is

$$U(\omega, 0) = F(\omega), \qquad \text{and} \qquad \frac{\partial U}{\partial t}(x, 0) = 0.$$

The solution for U is

$$U = a\cos(c\omega t) + b\sin(c\omega t),$$

and the condition $\partial U/\partial t = 0$ at $t = 0$ implies that $b = 0$. The condition that $U = F$ at $t = 0$ implies that $a = F$, and the solution is

$$U(\omega, t) = F(\omega)\cos(c\omega t).$$

3. We use the trigonometric identity to write

$$\cos(c\omega t) = \tfrac{1}{2}\left(\mathrm{e}^{\mathrm{i}c\omega t} + \mathrm{e}^{-\mathrm{i}c\omega t}\right).$$

We then have that

$$\begin{aligned} u(x,t) &= \int_{-\infty}^{\infty} U(\omega, t)\mathrm{e}^{-\mathrm{i}\omega x}\,\mathrm{d}\omega, && (10)\\ &= \tfrac{1}{2}\int_{-\infty}^{\infty} F(\omega)\left(\mathrm{e}^{-\mathrm{i}\omega x}\mathrm{e}^{-\mathrm{i}ct\omega} + \mathrm{e}^{-\mathrm{i}\omega x}\mathrm{e}^{+\mathrm{i}ct\omega}\right)\mathrm{d}\omega && (11)\\ &= \tfrac{1}{2}\int_{-\infty}^{\infty} F(\omega)\mathrm{e}^{-\mathrm{i}\omega x}\mathrm{e}^{-\mathrm{i}ct\omega}\,\mathrm{d}\omega + \tfrac{1}{2}\int_{-\infty}^{\infty} F(\omega)\mathrm{e}^{-\mathrm{i}\omega x}\mathrm{e}^{+\mathrm{i}ct\omega}\,\mathrm{d}\omega, && (12)\\ &= \tfrac{1}{2}f(x-ct) + \tfrac{1}{2}f(x+ct). && (13) \end{aligned}$$

In the final step, we have used the shift theorem to evaluate the inverse transforms. We could have also define intermediate variables $\xi = x - ct$ and $\eta = x + ct$ to do this job.

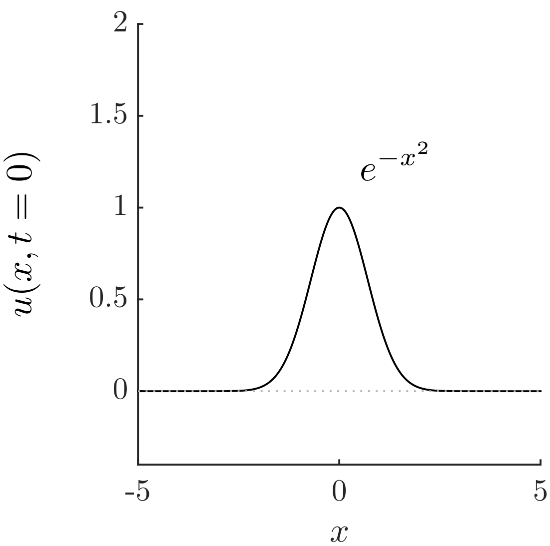
4 Method of characteristics. Consider the wave equation

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2},$$

with the initial conditions

$$u(x,0) = \mathrm{e}^{-x^2}, \qquad \text{and} \qquad \frac{\partial u}{\partial t} = 0.$$

The initial condition is given below.



Obtain the solution and sketch it at two later times.

Solution. The general solution to the wave equation obtained using the Method of Characteristics is

$$u(x,t) = \tfrac{1}{2} f(x+ct) + \tfrac{1}{2} f(x-ct) + \int_{x-ct}^{x+ct} g(\xi)\,\mathrm{d}\xi,$$

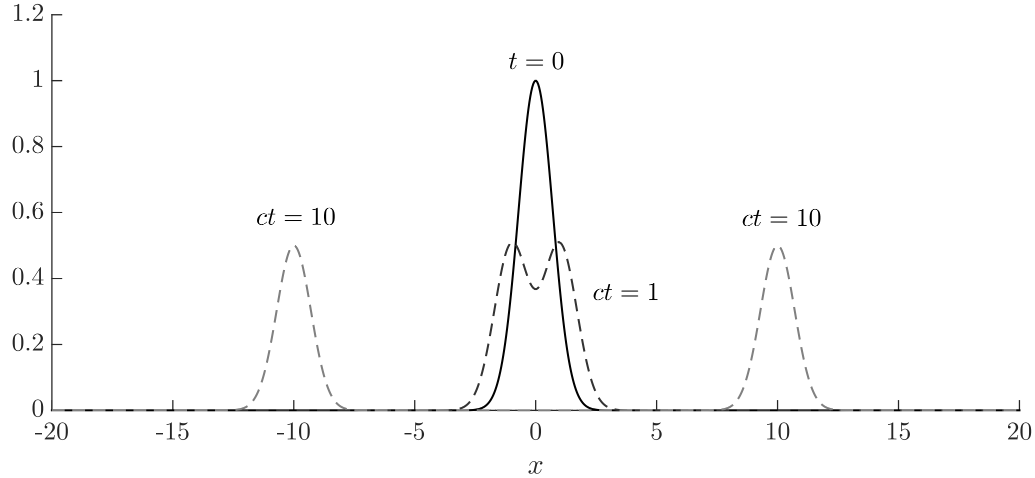
where $f(x)$ and $g(x)$ are given by the initial conditions,

$$u(x,t=0) = f(x), \qquad \text{and} \qquad \frac{\partial u}{\partial x}(x,t=0) = g(x).$$

Here, $g = 0$, and the solution is just

$$u(x,t) = \tfrac{1}{2}\mathrm{e}^{-(x-ct)^2} + \tfrac{1}{2}\mathrm{e}^{-(x+ct)^2}.$$

The solution consists of two Gaussian pulses traveling to the left and right. The solution is plotted at two later times below.



5 The wave equation in spherical polar coordinates. Consider the wave equation

$$\frac{\partial^2 u}{\partial t^2} = c^2 \nabla^2 u \,,$$

in three dimensions when the solution is spherically symmetric, so that

$$\nabla^2 u = \frac{1}{\rho^2} \frac{\partial}{\partial \rho} \left(\rho^2 \frac{\partial u}{\partial \rho} \right) \,.$$

Answer the following:

(a) Write $u(\rho, t) = \rho^{-1} w(\rho, t)$ and show that $w(\rho, t)$ obeys the one-dimensional wave equation.

(b) Hence solve for $u(\rho, t)$ in the general case.

(c) Explain why the resulting solution corresponds to one outgoing and one incoming wave.

Solution.

1. With $u = \rho^{-1} w$, we find

$$\frac{\partial u}{\partial \rho} = -\rho^{-2} w + \rho^{-1} \frac{\partial w}{\partial \rho} \,.$$

We therefore find that

$$\frac{1}{\rho^2} \frac{\partial}{\partial \rho} \left(\rho^2 \frac{\partial u}{\partial \rho} \right) = \frac{1}{\rho^2} \frac{\partial}{\partial \rho} \left(\rho^2 \left[-\rho^{-2} w + \rho^{-1} \frac{\partial w}{\partial \rho} \right] \right) \,, \tag{14}$$

$$= \frac{1}{\rho^2} \frac{\partial}{\partial \rho} \left(-w + \rho \frac{\partial w}{\partial \rho} \right) \,, \tag{15}$$

$$= \frac{1}{\rho^2} \left(-\frac{\partial w}{\partial \rho} + \frac{\partial w}{\partial \rho} + \rho \frac{\partial^2 w}{\partial \rho^2} \right) \,, \tag{16}$$

$$= \frac{1}{\rho} \frac{\partial^2 w}{\partial \rho^2} \,. \tag{17}$$

Meanwhile, because

$$\frac{\partial^2 u}{\partial t^2} = \frac{1}{\rho}\frac{\partial^2 w}{\partial t^2}\,,$$

we find that because u satisfies

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{1}{\rho^2}\frac{\partial}{\partial \rho}\left(\rho^2 \frac{\partial u}{\partial \rho}\right)\,,$$

w must in turn satisfy

$$\frac{\partial^2 w}{\partial t^2} = c^2 \frac{\partial^2 w}{\partial \rho^2}\,.$$

2. It turns out that w satisfies the ordinary wave equation, and we know the solution to the wave equation – it's just

$$w(\rho, t) = F(\rho + ct) + G(\rho - ct)\,,$$

where F and G are arbitrary functions determined by the initial conditions. Therefore $u(\rho, t) = \rho^{-1} w(\rho, t)$ is given by

$$u(\rho, t) = \rho^{-1} F(\rho + ct) + \rho^{-1} G(\rho - ct)\,,$$

where, again, F and G are arbitrary functions. In fact, if we had the conditions

$$u(\rho, 0) = f(\rho)\,, \qquad \text{and} \qquad \frac{\partial u}{\partial t}(\rho, 0) = g(\rho)\,,$$

then we have for w that

$$w(\rho, 0) = \rho f(\rho)\,, \qquad \text{and} \qquad \frac{\partial w}{\partial t}(\rho, 0) = \rho g(\rho)\,,$$

Thus the general solution for w is

$$w(\rho, t) = \tfrac{1}{2}(\rho + ct) f(\rho + ct) + \tfrac{1}{2}(\rho - ct) f(\rho - ct) + \int_{\rho - ct}^{\rho + ct} \varphi g(\varphi)\, \mathrm{d}\varphi\,,$$

and therefore the general solution for u is

$$u(\rho, t) = \tfrac{1}{2}(1 + \rho^{-1} ct) f(\rho + ct) + \tfrac{1}{2}(1 - \rho^{-1} ct) f(\rho - ct) + \rho^{-1} \int_{\rho - ct}^{\rho + ct} \varphi g(\varphi)\, \mathrm{d}\varphi\,,$$

3. It is easy to see from the form of the solution given by

$$u(\rho, t) = \rho^{-1} F(\rho + ct) + \rho^{-1} G(\rho - ct)\,,$$

that u consists of an "outgoing" wave – the G-part – and an "incoming wave", the F-part. The reason why G is outgoing is because as t increases, you're going to find the points at $G(x)$ are found at larger and larger ρ. So G is translating away from the center of the sphere. F is doing just the opposite, and is propagating inwards. In addition to this propagation, the dependence on ρ^{-1} means that both solutions are dilating and contracting according to the radius at which they are found.

A The infinite integral of the Gaussian

It should be well known to everyone that

$$\int_{-\infty}^{\infty} \mathrm{e}^{-x^2}\,\mathrm{d}x = \sqrt{\pi}\,.$$

Note also that because e^{-x^2} is symmetric about $x = 0$, this also implies that

$$\int_{0}^{\infty} \mathrm{e}^{-x^2}\,\mathrm{d}x = \tfrac{1}{2}\sqrt{\pi}\,.$$

This amazing fact is not at all obvious. We can prove it with a series steps involving the square of an integral, which we then reinterpret as an integral over two-dimensional Cartesian space, and then change into polar coordinates, which allows us to compute the answer. Here they are:

$$\begin{aligned}
I &= \left(\int_{-\infty}^{\infty} \mathrm{e}^{-x^2}\,\mathrm{d}x\right)^2, && (18)\\
&= \left(\int_{-\infty}^{\infty} \mathrm{e}^{-y^2}\,\mathrm{d}y\right)\left(\int_{-\infty}^{\infty} \mathrm{e}^{-x^2}\,\mathrm{d}x\right) && (19)\\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \mathrm{e}^{-x^2-y^2}\,\mathrm{d}x\,\mathrm{d}y\,, && (20)\\
&= 2\pi\int_{0}^{\infty} \mathrm{e}^{-r^2} r\,\mathrm{d}r\,, && (21)\\
&= 2\pi\Big[-\tfrac{1}{2}\mathrm{e}^{-r^2}\Big]_0^{\infty} && (22)\\
&= \pi\,. && (23)
\end{aligned}$$

In one of the intermediate steps we have converted Cartesian coordinates in (x, y) to polar coordinates in (r, θ), and evaluated the integral over θ (which is easy). The integral over r can then be calculated because it is an exact derivative. Finally, we then notice that $\int_{-\infty}^{\infty} \mathrm{e}^{-x^2}\,\mathrm{d}x = \sqrt{I} = \sqrt{\pi}$.