Quiz I

1 (6 points) Trigonometric integrals. Find I_1 and I_2.

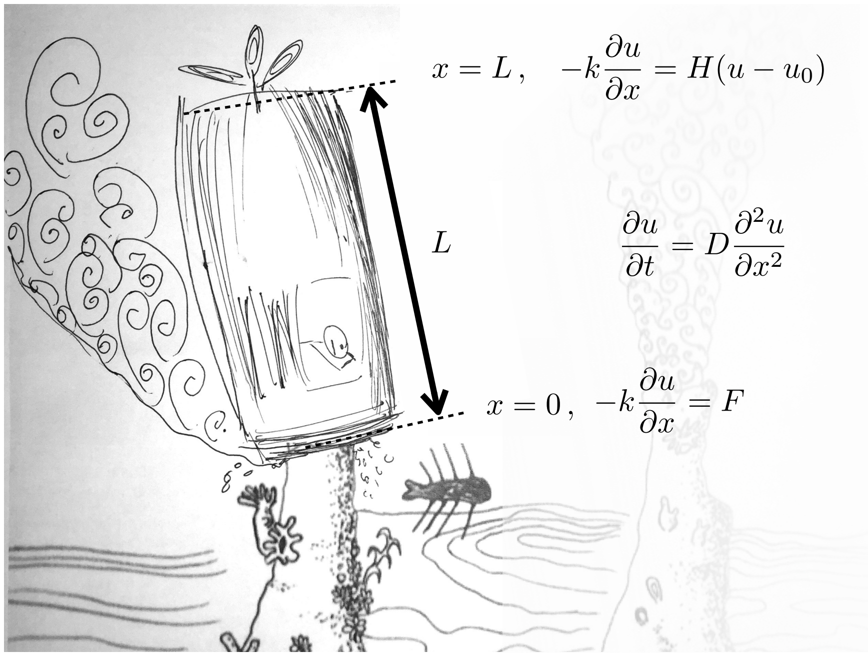
i) $$I_1 = \int_0^L \sin\left(\frac{\pi x}{L}\right)\cos\left(\frac{3\pi x}{L}\right)\,\mathrm{d}x\,.$$

ii) $$I_2 = \int_0^L \cos^2\left(\frac{6\pi x}{L}\right)\,\mathrm{d}x\,.$$

You may find the following results useful:

$$\sin(a)\cos(b) = \tfrac{1}{2}\left(\sin(a+b) + \sin(a-b)\right),$$
$$\cos(a)\cos(b) = \tfrac{1}{2}\left(\cos(a+b) + \cos(a-b)\right).$$

2 (9 points) Steady states. A submersible crashes into a hydrothermal vent filled with superheated water at the bottom of the ocean. A sketch of this scenario is given below.



We model the resulting temperature distribution $T(x, t)$ in the submersible using the one-dimensional heat conduction equation, taking the submersible as a metal cylinder with constant area A, length L, heat conductivity k and thermal diffusivity D. At $x = 0$ we assume a fixed temperature flux from the superheated water into the submersible, so that

$$-k\frac{\partial u}{\partial x} = F \qquad \text{at} \qquad x = 0\,.$$

The other end of the submersible, at $x = L$, is cooled by the ocean. We model the boundary at $x = L$ with the "Newton cooling" boundary condition, which means that

$$-k\frac{\partial u}{\partial x} = H\,(u - u_0) \qquad \text{at} \qquad x = L\,,$$

where u_0 is the temperature of the surrounding ocean. Answer the following.

1. What are the units of H? Assume u is in K, x is in meters, and k is in $WK^{-1}m^{-1}$. Explain why $H > 0$.

2. Assume that $\partial u/\partial t = 0$ (the submersible's temperature is not changing in time). What is the temperature distribution in the submersible? Your answer should give the temperature $u(x)$ as a function of x, in terms of the problem parameters (A, L, k, D, F, u_0 and H).

3. What happens to your solution as $H \to 0$? Why? *Hint: which of the preceding assumptions fails?*

3 (12 points) Separation of variables. Consider the time-dependent, one-dimensional heat conduction equation

$$\frac{\partial u}{\partial t} = D\frac{\partial^2 u}{\partial x^2},$$

with boundary conditions

$$u(x=0,t) = 0, \qquad \text{and} \qquad \frac{\partial u}{\partial x}(x=L,t) = 0$$

and the initial condition

$$u(x,t=0) = u_0 \sin\left(\frac{\pi x}{L}\right).$$

1. What is the "steady-state" or "equilibrium" distribution $u(x)$ when $\partial u/\partial t = 0$?

2. Substitute $u(x,t) = f(x)g(t)$ into the governing equation and derive two ordinary differential equations for $f(x)$ and $g(t)$ in terms of a "separation constant" λ.

3. Write down the general solution for $g(t)$.

4. Given that $u(x,t) = f(x)g(t)$, and given the boundary conditions on $u(x,t)$, deduce the boundary conditions for $f(x)$ at $x = 0$ and $x = L$.

5. (*) By solving the ODE for $f(x)$ with the correct boundary conditions, find the eigenvalues λ. Write your answer in terms of an integer n. Can you now use the initial condition to find the general solution for $u(x,t)$?