Quiz I

1 Trigonometric integrals.

i) Using the given identity and the fact that $\sin(-a) = -\sin(a)$, we find

$$\sin\left(\frac{\pi x}{L}\right)\cos\left(\frac{3\pi x}{L}\right) = \tfrac{1}{2}\sin\left(\frac{4\pi x}{L}\right) + \tfrac{1}{2}\sin\left(-\frac{2\pi x}{L}\right) ,$$
$$= \tfrac{1}{2}\sin\left(\frac{4\pi x}{L}\right) - \tfrac{1}{2}\sin\left(\frac{2\pi x}{L}\right) .$$

The integral I_1 is then

$$\begin{aligned} I_1 &= \int_0^L \sin\left(\frac{\pi x}{L}\right)\cos\left(\frac{3\pi x}{L}\right)\,\mathrm{d}x\,, \\ &= \frac{1}{2}\int_0^L \sin\left(\frac{4\pi x}{L}\right) - \sin\left(\frac{2\pi x}{L}\right)\,\mathrm{d}x\,, \\ &= \frac{1}{2}\left[-\frac{L}{4\pi}\cos\left(\frac{4\pi x}{L}\right) + \frac{L}{\pi}\cos\left(\frac{2\pi x}{L}\right)\right]_0^L , \\ &= 0\,. \end{aligned}$$

ii) Using the given identity, we find

$$\cos^2\left(\frac{6\pi x}{L}\right) = \tfrac{1}{2} + \tfrac{1}{2}\cos\left(\frac{12\pi x}{L}\right) .$$

The integral I_2 therefore becomes

$$\begin{aligned} I_2 &= \int_0^L \cos^2\left(\frac{6\pi x}{L}\right)\,\mathrm{d}x\,, \\ &= \frac{1}{2}\int_0^L 1 + \cos\left(\frac{12\pi x}{L}\right)\,\mathrm{d}x\,, \\ &= \frac{1}{2}\left[x + \frac{L}{12\pi}\sin\left(\frac{12\pi x}{L}\right)\right]_0^L , \\ &= \frac{L}{2}\,. \end{aligned}$$

2 Steady states.

1. The units of H can be found from the boundary condition. We have

$$H = \frac{-k\frac{\partial u}{\partial x}}{u-u_0},$$
$$= \left[\frac{\mathrm{WK^{-1}m^{-1}} \times \mathrm{Km^{-1}}}{\mathrm{K}}\right],$$
$$= \left[\frac{\mathrm{W}}{\mathrm{m^2K}}\right].$$

   The units of H are Watts per degree Kelvin-meter^2, since it is a ratio between a temperature difference in Kelvin, and heat flux, which has units Watts per meter^2. $H > 0$ is required by physics: it means that a positive temperature difference between the end of the submersible at $x = L$, and the surrounding ocean, which is at $x > L$, is associated with a flux of heat in the positive x-direction (recall heat flux is $-k\frac{\partial u}{\partial x}$).

   Another way to think about this is through a thought experiment. When u_0 is very cold, we expect that the temperature u is *decreasing* as x approaches L. If u is decreasing as x increases, this means $\partial u/\partial x < 0$. Thus, both $-k\partial u/\partial x$ and $u - u_0$ are positive, which implies that H is positive as well.

   Note that this argument depends on the fact that, at $x = L$, a flux out of the submersible implies that$-k\partial u/\partial x > 0$. The opposite is true for the end at $x = 0$.

2. When u is not a function of t, the heat equation reduces to

$$0 = \frac{\mathrm{d}^2u}{\mathrm{d}x^2}.$$

   The solution is

$$u = Ax + B,$$

   where A and B are undetermined constants. Note that $\mathrm{d}u/\mathrm{d}x = A$. This enables us to apply the boundary condition at $x = 0$, giving

$$-\frac{F}{k} = A.$$

   Therefore $u = -Fx/k + B$. We use this to calculate u at $x = L$, and find

$$u(x = L) = -\frac{FL}{k} + B.$$

   Putting this into the boundary condition at $x = L$ implies

$$F = H\left(-\frac{FL}{k} + B - u_0\right),$$

and solving for B yields

$$B = \frac{F}{H} + u_0 + \frac{FL}{k}\,.$$

The total solution is then

$$u(x) = \frac{F}{k}\,(L - x) + u_0 + \frac{F}{H}\,.$$

3. As $H \to 0$, we find that the temperature blows up, or that $u \to \infty$. This is unphysical. Recall that to find the above solution, we assumed that u was not a function of t – in other words, we assumed that a steady-state solution exists. However, when $H \to 0$, the boundary condition at $x = L$ becomes that of an insulator, or limits to

$$-k\frac{\partial u}{\partial x} = 0\,.$$

In this case, we are fluxing heat in at $x = 0$, but not allowing it to escape at $x = L$. The consequence is that the solution increases in time, and there is no steady solution.

3 Separation of variables.

1. The steady-state solution is $T = 0$.

2. Substituting $u(x,t) = f(x)g(t)$ into the governing equation yields

$$f\frac{\mathrm{d}g}{\mathrm{d}t} = Dg\frac{\mathrm{d}^2 f}{\mathrm{d}x^2}\,.$$

Next we divide by Dfg. This yields

$$\underbrace{\frac{1}{Dg}\frac{\mathrm{d}g}{\mathrm{d}t}}_{\text{function of } t \text{ only}} \quad = \quad \underbrace{\frac{1}{f}\frac{\mathrm{d}^2 f}{\mathrm{d}x^2}}_{\text{function of } x \text{ only}} \quad = -\lambda\,.$$

Since both right and left are functions of either t or x, they cannot be equal to each other unless they equal a constant, which we denote $-\lambda$. We then obtain the ODEs

$$\frac{1}{Dg}\frac{\mathrm{d}g}{\mathrm{d}t} = -\lambda \qquad \Longrightarrow \qquad \frac{\mathrm{d}g}{\mathrm{d}t} + \lambda D g = 0\,,$$

and

$$\frac{1}{f}\frac{\mathrm{d}^2 f}{\mathrm{d}x^2} = -\lambda \qquad \Longrightarrow \qquad \frac{\mathrm{d}^2 f}{\mathrm{d}x^2} + \lambda f = 0\,.$$

3. The function $g(t)$ obeys
$$\frac{\mathrm{d}g}{\mathrm{d}t} + \lambda D g = 0\,.$$
This is a linear first-order ordinary differential equation, with exponential solutions. The solution is
$$g(t) = A\mathrm{e}^{-\lambda D t}\,,$$
where both A and λ are undetermined at this point.

4. The boundary conditions on $f(x)$ follow from the boundary conditions on $u(x,t)$. The boundary condition at $x = 0$ is
$$u(x=0,t) = f(x=0)g(t) = 0\,.$$
Since $g(t)$ is not zero in general, this implies that $f(x=0) = 0$. At $x = L$ we have
$$\frac{\partial u}{\partial x}(x=L,t) = g(t)\frac{\mathrm{d}f}{\mathrm{d}x}(x=0) = 0\,.$$
Again, $g(t)$ does not equal zero in general, so $\mathrm{d}f/\mathrm{d}x(x=L) = 0$. In summary, the two conditions on $f(x)$ are
$$f(x=0) = 0 \qquad \text{and} \qquad \frac{\mathrm{d}f}{\mathrm{d}x}(x=L) = 0\,.$$

5. The function $f(x)$ obeys the equation
$$\frac{\mathrm{d}^2 f}{\mathrm{d}x^2} + \lambda f = 0\,,$$
with boundary conditions given above. If $\lambda \leq 0$, the only solution is $f = 0$, the trivial solution. This does not satisfy the initial condition, so we must have $\lambda > 0$. In this case the solutions can be written
$$f(x) = B\sin\left(\sqrt{\lambda}x\right) + C\cos\left(\sqrt{\lambda}x\right)\,.$$
The condition $f(x=0) = 0$ implies that $C = 0$. The condition at $x = L$ implies that
$$\sqrt{\lambda}B\cos\left(\sqrt{\lambda}L\right) = 0\,.$$
Either $\lambda = 0$ or $B = 0$ implies that $f = 0$, the trivial solution. Thus we conclude that λ must take values such that
$$\cos\left(\sqrt{\lambda}L\right) = 0\,.$$
This happens with
$$\sqrt{\lambda}L = \pi\left(n - \tfrac{1}{2}\right)\,, \qquad \text{for} \qquad n = 1, 2, 3, \cdots\,,$$

which implies that

$$\lambda_n = \left[\frac{\pi}{L}\left(n - \tfrac{1}{2}\right)\right]^2 \qquad \text{for} \qquad n = 1, 2, 3, \cdots .$$

To solve the initial condition, we note that there are an infinite number of solutions, each corresponding to a particular value of λ_n, and thus a particular $u_n = f_n g_n$. The general solution is the sum of all these solutions, which is

$$u(x,t) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{\pi}{L}(n - \tfrac{1}{2})x\right) \exp\left\{-D\left(\frac{\pi}{L}(n - \tfrac{1}{2})\right) t\right\} .$$

To find the coefficients B_n, we use the initial condition. Taking $t = 0$, the initial condition implies that

$$u_0 \sin\left(\frac{\pi x}{L}\right) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{\pi}{L}(n - \tfrac{1}{2})x\right) .$$

The coefficients B_n are then found by multiplying this equation by

$$\sin\left(\frac{\pi}{L}(m - \tfrac{1}{2})x\right) ,$$

and integrating from 0 to L. Because of the properties of trigonometric integrals, all of the terms in the sum integrate to 0, except one, when $n = m$. We thus find that

$$B_m = \frac{2}{L}\int_0^L u_0 \sin\left(\frac{\pi x}{L}\right) \sin\left(\frac{\pi}{L}(m - \tfrac{1}{2})x\right) .$$

This is the formula for the coefficients B_m. Using the trigonometric identity

$$\sin(a)\sin(b) = \tfrac{1}{2}\left[\cos(a-b) - \cos(a+b)\right] ,$$

we find

$$B_m = \frac{u_0(-1)^m}{\pi}\left(\frac{1}{m - 3/2} + \frac{1}{m + 1/2}\right) .$$

The final solution is then

$$u(x,t) = \frac{u_0}{\pi}\sum_{n=1}^{\infty}\left(\frac{(-1)^m}{m - 3/2} + \frac{(-1)^m}{m + 1/2}\right) \sin\left(\frac{\pi}{L}(n - \tfrac{1}{2})x\right) \exp\left\{-D\left(\frac{\pi}{L}(n - \tfrac{1}{2})\right) t\right\} .$$