Quiz 2

1 Fourier series (10 points)

(a) From the expression for the exponential form of the Fourier series over the interval $(-\pi, \pi)$

$$f(x) = \sum_{n=-\infty}^{\infty} c_n \mathrm{e}^{\mathrm{i}nx},$$

find the expression for c_n as an integral involving $f(x)$.

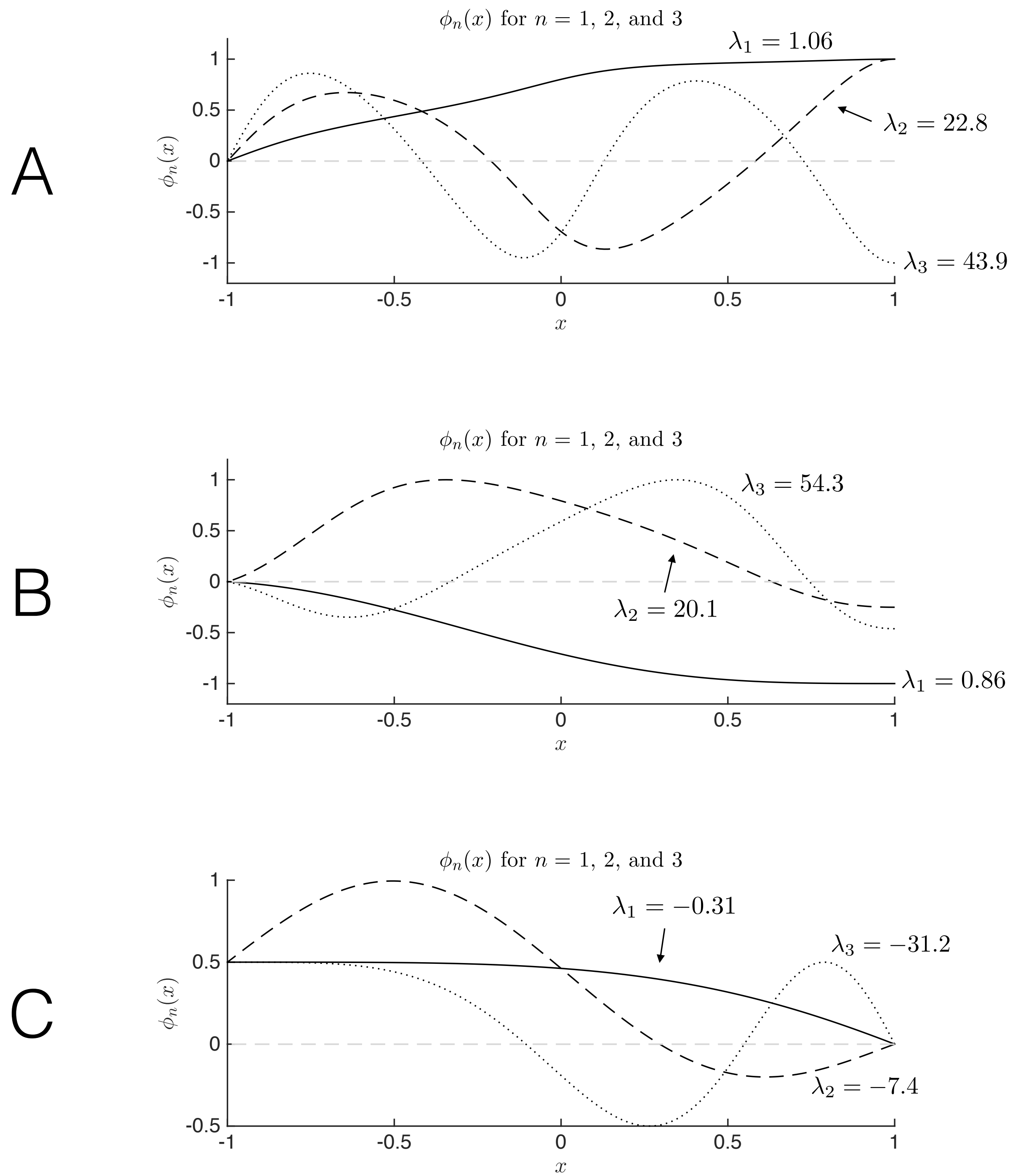
(b) Now consider the following function:

$$f(x) = \begin{cases} 1 & \text{for} \quad -\pi < x < 0, \\ e^{-ax} & \quad \text{for} \quad 0 < x < \pi\,. \end{cases}$$

Find the complex coefficients c_n for the Fourier series of $f(x)$.

(c) Explain what happens in the limits $a \to 0$ and $a \to \infty$.

2 Sturm–Liouville theory (5 points). The below figure plots the first three eigenfunctions which solve differential equations "A", "B", and "C". From the form of the eigenfunctions and the eigenvalues, determine whether the underlying differential equation can be a regular Sturm–Liouville problem. Give your reasons (one line per case should be enough).



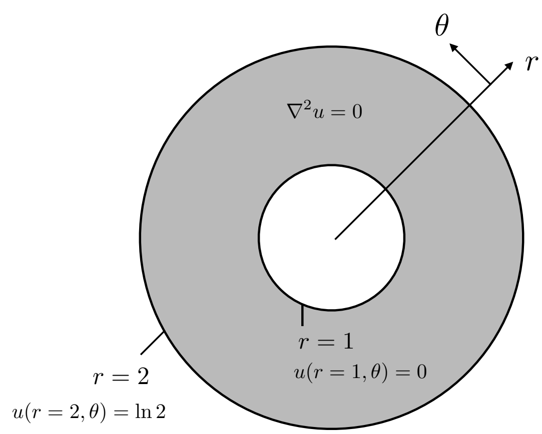
Problem 2. Eigenfunctions for three differential equations labeled A, B, and C.

3 Laplace in an annulus (10 points). Consider Laplace's equation in an annulus. The annulus has an inner radius of 1 and and outer radius of 2; a sketch of the annulus domain is given below. Laplace's equation in polar coordinates is

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0\,.$$

The boundary conditions are

$$u(r=1,\theta) = 0 \qquad \text{and} \qquad u(r=2,\theta) = \ln 2\,.$$



Answer the following:

(a) Use separation of variables to find the general solution $u(r, \theta)$.

(b) Find the solution which satisfies the boundary conditions at $r = 1$ and $r = 2$.

(c) If u is temperature, the total heat flux flowing *toward the origin* at radius r is

$$Q(r) = \int_0^{2\pi} q(r,\theta) r \, \mathrm{d}\theta\,,$$

where

$$q(r,\theta) = k\frac{\partial u}{\partial r}$$

is the inward heat flux density (the flux in the negative r-direction). Determine $Q(r = 2)$ and $Q(r = 1)$. What do you observe? How could you have predicted this from the governing problem without computing the integrals?