Quiz 2

1 Fourier series (10 points)

(a) From the expression for the exponential form of the Fourier series over the interval $(-\pi, \pi)$

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{\mathrm{i}nx} ,$$

find the expression for c_n as an integral involving $f(x)$.

(b) Now consider the following function:

$$f(x) = \begin{cases} 1 & \text{for} \quad -\pi < x < 0, \\ e^{-ax} & \text{for} \quad 0 < x < \pi . \end{cases}$$

Find the complex coefficients c_n for the Fourier series of $f(x)$.

Solution. The c_n are given by

$$c_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{-\mathrm{i}nx} \, \mathrm{d}x .$$

Therefore for $n = 0$ we have

$$\begin{aligned} c_0 &= \frac{1}{2\pi} \left[\int_{-\pi}^{0} 1 \, \mathrm{d}x + \int_{0}^{\pi} e^{-ax} \, \mathrm{d}x \right], \\ &= \frac{1}{2} + \frac{1 - e^{-a\pi}}{2\pi a} . \end{aligned}$$

and for $n > 1$,

$$\begin{aligned} c_n &= \frac{1}{2\pi} \int_{-\pi}^{0} e^{-\mathrm{i}nx} \, \mathrm{d}x + \frac{1}{2\pi} \int_{0}^{\pi} e^{-(\mathrm{i}n+a)x} \, \mathrm{d}x , \\ &= \frac{1}{2\pi} \left[- \frac{e^{-\mathrm{i}nx}}{\mathrm{i}n} \Big|_{-\pi}^{0} - \frac{e^{-(\mathrm{i}n+a)x}}{\mathrm{i}n + a} \Big|_{0}^{\pi} \right] \\ &= \frac{1}{2\pi} \left[\frac{e^{\mathrm{i}n\pi} - 1}{\mathrm{i}n} + \frac{1 - e^{-\mathrm{i}n\pi} e^{-a\pi}}{\mathrm{i}n + a} \right], \\ &= \frac{(-1)^n - 1}{2\mathrm{i}\pi n} - \frac{(-1)^n e^{-a\pi} - 1}{2\pi(\mathrm{i}n + a)} . \end{aligned}$$

(c) Explain what happens in the limits $a \to 0$ and $a \to \infty$.

Solution. For $a \to 0$, we can simply plug $a = 0$ into c_n to find that $c_n \to 0$ for $n > 0$. Finding c_0 requires more care. Note that e^y has the Taylor expansion around $y = 0$,

$$\mathrm{e}^y = 1 + y + \tfrac{1}{2}y^2 + O(y^3)\,.$$

Therefore

$$\begin{aligned}\lim_{a\to 0} c_0 &= \lim_{a\to 0}\left(\frac{1}{2} + \frac{1}{2\pi a}\left[1 - (1 - a\pi + \tfrac{1}{2}(a\pi)^2 + \cdots)\right]\right),\\ &= \lim_{a\to 0}\left(\frac{1}{2} + \frac{1}{2} - \frac{\pi a}{4} + \cdots\right),\\ &= 1\,.\end{aligned}$$

This limit corresponds to $f(x)$ being a straight line, so it makes sense that $c_0 = 1$ and all other $c_n = 0$ – and an alternative way to obtain the Fourier coefficients is simply to plug $a = 0$ into the form for $f(x)$.

The limit $a \to \infty$ is more straightforward and follows immediately. We find

$$c_n \to \frac{(-1)^n - 1}{2\mathrm{i}\pi n}\,,$$

and $c_0 \to 1/2$. This limit corresponds to $f(x)$ becoming a square pulse.

2 Sturm–Liouville theory (5 points). The below figure plots the first three eigenfunctions which solve differential equations "A", "B", and "C". From the form of the eigenfunctions, determine whether the associated differential equation is a regular Sturm–Liouville problem. Give your reasons (one line per case is enough).

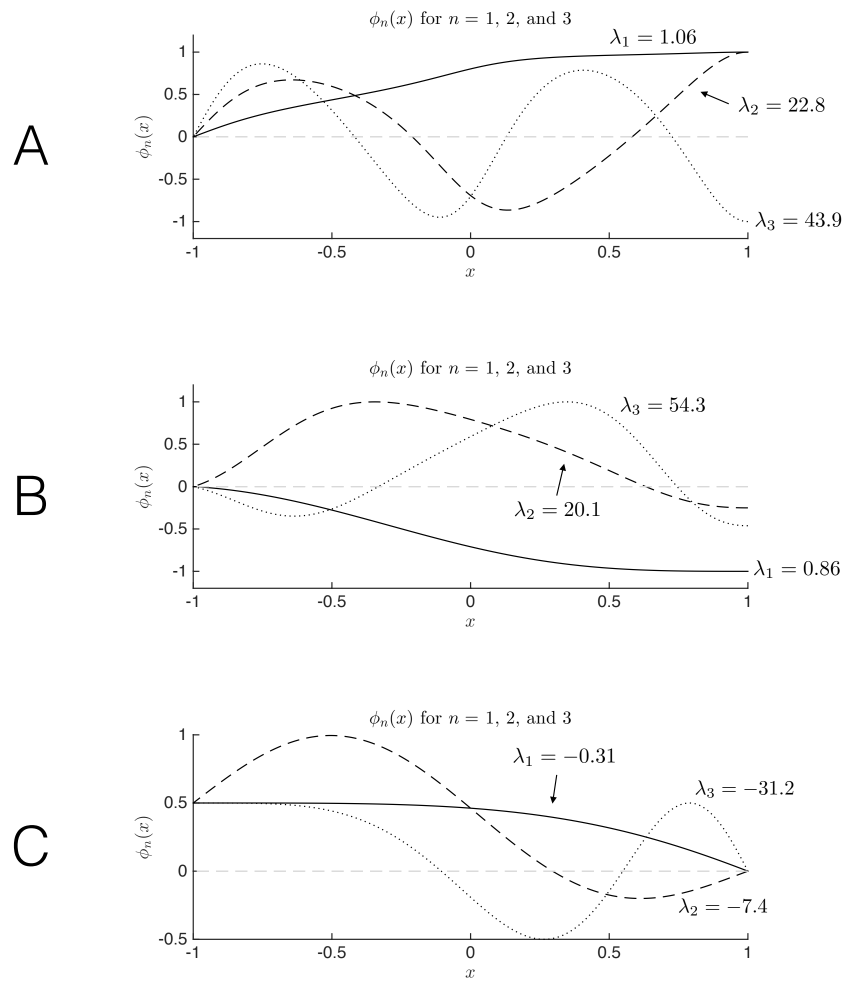


Figure 1: Problem 2. Eigenfunctions for three differential equations labeled A, B, and C.

Solution.

A. Not Sturm-Liouville. The function corresponding to λ_2 has two zeros and the one corresponding to λ_3 has three. Sturm-Liouville theory predicts that the n^{th} mode has $n - 1$ zeros, thus second and third modes must have one and two zeros, respectively.

B. These are legitimate Sturm-Liouville eigenfunctions.

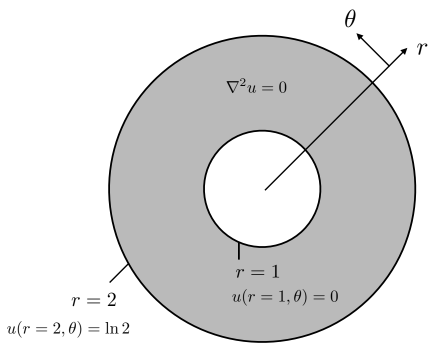
C. Not Sturm-Liouville. The eigenvalues have the wrong behavior, decreasing as the mode number increases. Also, the left-hand boundary condition is inhomogeneous, as $\mathrm{d}\phi/\mathrm{d}x + \beta u$ cannot possibly equal 0 for any β for all three functions.

3 Laplace in an annulus (10 points). Consider Laplace's equation in an annulus. The annulus has an inner radius of 1 and and outer radius of 2; a sketch of the annulus domain is given in figure XX. Laplace's equation in polar coordinates is

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}=0\,.$$

The boundary conditions are

$$u(r=1,\theta)=0 \qquad \text{and} \qquad u(r=2,\theta)=\ln 2\,.$$



Answer the following:

(a) Use separation of variables to find the general solution $u(r,\theta)$.

(b) Find the solution which satisfies the boundary conditions at $r=1$ and $r=2$.

(c) If u is temperature, the total heat flux flowing *toward the origin* at radius r is

$$Q(r)=\int_0^{2\pi} q(r,\theta)r\,\mathrm{d}\theta\,,$$

where

$$q(r,\theta)=k\frac{\partial u}{\partial r}$$

is the inward heat flux density (the flux in the negative r-direction). Determine $Q(r=2)$ and $Q(r=1)$. What do you observe? How could you have predicted this from the governing problem without computing the integrals?

Solution.

(a) To separate variables we propose $u(r,\theta)=f(r)g(\theta)$ and plug this into Laplace's equation. This yields

$$\frac{g}{f}\left(rf'\right)'+\frac{f}{r^2}g''=0\,.$$

To isolate terms dependent on r and θ respectively, we multiply by r^2/fg and move the g-terms to the other side of the equation. This yields

$$\frac{r}{f}\left(rf'\right)' = -\frac{g''}{g} = \lambda\,,$$

where we have defined a separation constant λ. The θ-equation is

$$g'' + \lambda g = 0\,.$$

$g(\theta)$ must be periodic such that $g(0) = g(2\pi)$ and $g'(0) = g'(2\pi)$. The solution is therefore

$$g = A\sin(n\theta) + B\cos(n\theta)\,,$$

where we have determined the eigenvalue $n = \sqrt{\lambda} = 0, 1, 2, 3....$ Here, we choose $n \geq 0$ for simplicity; we must choose either $n \geq 0$ or $n \leq 0$. The r-equation is then

$$r^2 f'' + rf' - n^2 f = 0\,.$$

To solve this for $n > 0$, we propose $f_n = Cr^\alpha$, to yield an equation for α,

$$\alpha^2 = n^2\,,$$

which implies $\alpha = \pm n$. Thus $f_n(r)$ is

$$f_n(r) = Cr^n + Dr^{-n}\,.$$

The condition at $r = 1$ implies that $C + D = 0$, or that $D = -C$, and

$$f_n(r) = C\left(r^n - r^{-n}\right)\,.$$

When $n = 0$, this form only gives one of the solutions for $f(r)$. To find the other solution, we return to the r-equation for $n = 0$,

$$r^2 f_0'' + rf_0' = 0\,.$$

This is a first-order equation for f', whose solution is $f_0' = E/r$. We can thus integrate to find f_0:

$$f_0(r) = E\ln r + F\,.$$

The condition at $r = 1$ implies that $F = 0$. Putting f and g together, and adding all the solutions for every n yields

$$u(r,\theta) = E\ln r + \sum_{n=1}^{\infty}\left(r^n - r^{-n}\right)\left[A_n\sin(n\theta) + B_n\cos(n\theta)\right]\,.$$

(b) The boundary condition at $r = 2$ is $u(r, \theta) = \ln 2$. Applying the boundary condition implies that

$$\ln 2 = E \ln 2 + \sum_{n=1}^{\infty} \left(2^n - 2^{-n}\right) \left[A_n \sin(n\theta) + B_n \cos(n\theta)\right] .$$

First, we simply integrate this condition from $\theta = 0$ to $\theta = 2\pi$. This eliminates all the terms inside the summation and yields

$$\int_0^{2\pi} \ln 2 \, \mathrm{d}\theta = \int_0^{2\pi} E \ln 2 \, \mathrm{d}\theta , \qquad \text{which implies} \qquad E = 1 .$$

Now, if we multiply by either $\sin(m\theta)$ or $\cos(m\theta)$, the boundary condition on the left side disappears. Thus the solution is just

$$u(r, \theta) = \ln r .$$

This also follows from the fact that $u = \ln r$ satisfies the boundary conditions at $r = 1$ and $r = 2$ as well as the governing equation, and so must be the full solution.

(c) The origin-flowing heat flux density is

$$q(r, \theta) = k \frac{\partial u}{\partial r} = \frac{1}{r} .$$

Therefore, the total origin-flowing heat flux at $r = 1$ is

$$Q(1) = \int_0^{2\pi} k \, \mathrm{d}\theta = 2k\pi .$$

The total origin-flowing heat flux at $r = 2$, on the other hand, is

$$Q(2) = \int_0^{2\pi} \frac{k}{2} 2 \, \mathrm{d}\theta = 2k\pi .$$

They are equal. This must be true, because if they were not equal, the corresponding heat conduction problem would not have the steady-solution (which we were able to find).

An alternate, mathematical reason for this fact can be given by integrating Laplace's equation over the volume of the annulus. This would yield the fact that the total integral of $\hat{\mathbf{n}} \cdot \nabla u$ over the boundary of the annulus must be zero. Finally, notice that $Q(r = 2)$ is exactly that integral for the part of the boundary at $r = 2$, whereas $Q(r = 1)$ is the negative of the integral corresponding to the boundary at $r = 1$. Thus this fact implies that $Q(1) = Q(2)$.